Doctrinal and Groupoidal Representations of Classifying Topoi

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Abstract

We study two species of representing data for classifying topoi. First, we exposit an approach to classifying topos theory using Lawverian doctrines. Contributions are made to relative topos theory and internal locale theory in order to accommodate our doctrinal approach. Applications of our development are then made to the study of syntactic completions of doctrines. We also study the representation of classifying topoi by localic and topological groupoids, culminating in a model-theoretic characterisation of which open topological groupoids represent the classifying topos of a theory.

Contents

Introduction

Topoi (by topos unqualified, we mean Grothendieck topos) were originally introduced by Grothendieck to tackle problems arising in algebraic geometry and algebraic topology, but have since become central to the categorical study of logic. In this dissertation, we study two species of representing data for the *classifying topos* of a predicate (or first-order) theory – doctrinal and groupoidal representations:

The classifying topos of a theory,

the doctrine of a theory, **groupoids of models.**

Prior to motivating the use of these representing data, we recall some important examples of topoi and the notion of a classifying topos.

Topos theory and topology. Many natural examples of topoi are generated from topological starting data:

Examples 0.1. (i) (Sheaves on a space) Given a topological space X, the slice category $\mathbf{LH}/X$, where $\mathbf{LH} \subseteq \mathbf{Top}$ is the subcategory of local homeomorphisms between topological spaces, is a topos: the familiar topos $\mathbf{Sh}(X)$ of *sheaves* on X.

Topos theory subsumes the point-free incarnation of topology, *locale theory* (see [61], [97] or Section II.1), in that the map $X \mapsto \mathbf{Sh}(X)$ yields a fully faithful 2-embedding

$$\mathbf{Sh}\colon \mathbf{Loc} \hookrightarrow \mathbf{Topos}$$

of locales into the bicategory of topoi, geometric morphisms, and 2-cells between these (i.e. natural transformations between the inverse image functors).

(ii) (Topoi of continuous actions) If G is a topological group (or even a monoid), the category $\mathbf{B}G$ of continuous actions by G on discrete sets is also a topos.

There is a sense in which every topos is a generalisation of the notion of topological space in which ‘points can have non-trivial isomorphisms’. This is expressed by the representation results of [68] and [17] discussed in Part B. In this manner, topoi can be likened to orbifolds from differential geometry (cf. [94], [105]).

The notion of a ‘space whose points have isomorphisms’ is captured formally by an *internal groupoid* of either the category of topological spaces **Top**, or locales **Loc** (if ‘space’ is taken in the pointfree sense). Each localic or topological groupoid can be associated with a *topos of equivariant sheaves* that generalises both species of topoi from Examples 0.1 (full definitions are given in Chapter V). The representation results [17], [68] state that every topos (with enough points) is equivalent to the sheaves on an *open* localic/topological groupoid.

Classifying topoi. In addition to a topological description, each topos admits a logical representation via the notion of a *classifying topos*. The terminology 'classifying topos' was inspired by analogy with classifying spaces from algebraic topology. Indeed, the first examples of classifying topoi from [3, §IV.2.3-4], namely the topoi of the form $\mathbf{B}G$, play the same role as the classifying spaces of principal bundles (see, for instance, [90]).

The *classifying topos* of a theory $\mathbb{T}$ is a topos $\mathcal{E}_{\mathbb{T}}$ for which there is an equivalence

$$\mathbb{T}\text{-mod}(\mathcal{F}) \simeq \mathbf{Geom}(\mathcal{F}, \mathcal{E}_{\mathbb{T}})$$

natural in $\mathcal{F}$, where $\mathbb{T}\text{-mod}(\mathcal{F})$ is the category of models of $\mathbb{T}$ in the topos $\mathcal{F}$ (see [63, §D1] for how to construct models internal to an arbitrary topos) and $\mathbf{Geom}(\mathcal{F}, \mathcal{E}_{\mathbb{T}})$ is the category of geometric morphisms from $\mathcal{F}$ to $\mathcal{E}_{\mathbb{T}}$. By this universal property, the classifying topos of a theory is unique up to equivalence.

Example 0.2 (Remark D3.1.14 [63])**.** The Lindenbaum-Tarski algebra $L_{\mathbb{T}}$ of a propositional geometric theory $\mathbb{T}$ is a *locale*, i.e. a 'pointfree space', and the topos $\mathbf{Sh}(L_{\mathbb{T}})$ classifies $\mathbb{T}$.

Geometric logic is that fragment of infinitary first-order logic whose permissible symbols are equality $=$, truth $\top$, falsity $\bot$, finite conjunction $\wedge$, infinitary disjunction $\bigvee$ and existential quantification $\exists$. In [1, §2.3], Abramsky describes geometric logic as the logic of *observable properties* – those properties that can be determined to hold on the basis of a finite amount of information. Not only does every geometric theory have a classifying topos (see [22, Theorem 2.1.10]), but every topos is the classifying topos for some geometric theory (see [22, Theorem 2.1.11]). Classifying topos theory lends geometric logic a strong spatial intuition, as explored in [124]. Note, however, that theories from other fragments of predicate logic can also have classifying topoi. Intuitively, the classifying topos embodies the essential information about a theory.

(A) Doctrinal representations.

It is therefore of interest for the logician to study representations of classifying topoi, as these are effectively representations of the logical theory by other data. We first consider representations of topoi by *doctrines* in the sense of Lawvere [77]. Doctrine theory represents another approach to categorical logic, parallel to classifying topos theory. A doctrine is a categorical generalisation of the notion of a Lindenbaum-Tarski algebra to the first-order setting, an alternative to the cylindrical algebras suggested by Tarski [50] and polyadic algebras suggested by Halmos [47].

In Part A, we exposit a classifying topos theory for Lawverian doctrines. We will observe in Section III.4 that many previously known results in classifying topos theory admit intuitive proofs when phrased in the language of doctrine theory. Additionally, we compare our doctrinal construction of the classifying topos of a theory with the standard textbook account involving *syntactic categories*.

A topos-theoretic framework for completions of doctrines. Our classifying topos theory for doctrines also yields applications to doctrine theory. In recent years, many *syntacitc completions* of doctrines have been considered in the literature (e.g. [30], [96], [119], [121]). In Chapter IV, we introduce the *geometric completion* of a doctrine, and develop a topos-theoretic framework for generating completions of doctrines to other subfragments of geometric logic.

The geometric completion we introduce is *semantically invariant*, meaning the category of models associated with a doctrine and its geometric completion are equivalent. Thus, we intend to study the semantics for various doctrines with a unified approach using the familiar language of geometric logic.

(B) Groupoidal representations.

A classical result of model theory asserts that an $\aleph_0$-categorical theory is entirely determined, up to the level of *bi-interpretability*, by the topological automorphism group of its unique countable model (proven by Ahlbrandt and Ziegler in [2, §1], attributed to unpublished work of Coquand). Explicitly, given $\aleph_0$-categorical theories $\mathbb{T}_1$ and $\mathbb{T}_2$,

$$\mathbb{T}_1, \mathbb{T}_2 \text{ are bi-interpretable} \iff \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$$

where M and N are countable models of $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. As shown in recent work by Ben Yaacov [9], the assumption that the theories are $\aleph_0$-categorical can be removed by replacing a topological group with a topological groupoid (however, this groupoid is not a groupoid of models).

However, in many ways bi-interpretability is too fine an equivalence on first-order theories – there are theories that ought to be considered 'equivalent', but which are not bi-interpretable (see [7], [70, Example 8.4], [123, §4.7]). In the topos-theoretic approach to predicate logic, it is more natural to consider a strictly weaker notion of equivalence between theories (which is satisfied by the examples referenced above, see for example [122]).

Definition 0.3 (§D1.4.13 [63], §2.2.1 [22])**.** Given two theories $\mathbb{T}, \mathbb{T}'$ with classifying topoi, there is a (natural) equivalence

$$\mathbb{T}\text{-mod}(\mathcal{F}) \simeq \mathbb{T}'\text{-mod}(\mathcal{F})$$

of models, for each topos $\mathcal{F}$, if and only if there is an equivalence of topoi $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$. We call such equivalences *Morita equivalences*, after Morita equivalence for rings [95].

The representation of classifying topoi by topological groupoids. Part B of this dissertation concerns a study of Morita equivalence of first-order theories employing topological groupoids, paralleling the classical model theoretic account for bi-interpretability. We seek to characterise Morita equivalence of theories in terms of which open topological groupoids 'represent' a given theory in the following sense.

Definition 0.4. An (open) topological groupoid is said to *represent* a theory if its topos of sheaves classifies the theory.

Morita equivalence of theories can thus be translated into Morita equivalence of their representing groupoids, that is if their topoi of sheaves are equivalent. Already in the case of topological groups, the question of Morita equivalence is rather involved, as evidenced in [91].

Relation to Stone-type and Galois-type representations. As a 'generalised space in which points have isomorphisms', the classifying topos of a theory is the generalised space whose points are models of a theory, and whose isomorphisms are isomorphisms of these models. Thus, as explained in [5], [4, §5-6], we intuit that the representation of classifying topoi by a topological groupoid of models is a predicate extension of Stone duality for propositional theories [111], [112], which associates a theory of propositional logic to its *space of models* (this contrasts with the 'first-order duality' of Makkai [86, §5-8], [85] where a groupoid of models is equipped with *ultracategory* structure rather than topological structure).

Moreover, as formalised in [14, §7], the representation of topoi by localic/topological groupoids also represents an abstraction of Grothendieck's Galois theory (see [44, §V.4]). Thus, the representation of classifying topoi by a topological groupoid represents a common generalisation of Stone-type and Galois-type representations for logical theories.

Classification result and consequences. In Part B, we characterise which groupoids of models constitute a *representation* of the classifying topos of a theory, subsuming the previous topological representation results [5], [11], [17], [21], [36], [37] found in the literature. Intuitively, this expresses which groupoids of models 'have enough information to recover' the theory. Our characterisation has a distinctly model-theoretic flavour, contrasting with localic representation results of [34], [68].

Subsequently, we demonstrate that every geometric morphism between topoi with enough points is induced by a homomorphism of topological groupoids, and thereby establish a biequivalence for topoi with enough points, giving a topological parallel for the analogous result in the localic setting [92, §7]. Informally, this expresses that, just as topoi are 'spaces whose points have isomorphisms', geometric morphisms are 'continuous maps that respect isomorphisms between points'.

As a consequence, in Corollary VIII.15, we will observe that two theories with representing groupoids are Morita equivalent if and only if their representing groupoids can be compared via a cospan of *weak equivalences*. Thus, we have transformed the problem of Morita equivalence into the domain of topological algebra.

Chapter overview

Since, at times, we will detour through subjects that could seem distant to our original motivation, we include an overview motivating the content of each chapter as it relates to our overarching aims.

Chapter II is adapted from the preprint [127]; modified content from the preprint [128] is split across Chapter III and Chapter IV, while the preprint [126] has become Chapter V and Chapter VII; Chapter VI is taken from joint work with Graham Manuell in the preprint [88].

Relative topos theory

Relative topoi. Chapter I concerns *relative topos theory* in the sense of [26]. Because doctrines, in whose language we develop our logical applications, are examples of fibred categories, it is natural to seek a relative formalism. If topos theory is the study of the bicategory **Topos** of topoi, geometric morphisms, and natural transformations (between the inverse image functors), then relative topos theory is the study of the bicategory **RelTopos** of relative topoi,

(i) the bicategory whose objects are geometric morphisms $f\colon \mathcal{F} \to \mathcal{E}$,

(ii) whose arrows are squares of geometric morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[r, "k"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{E} \arrow[r, "h"'] & \mathcal{H} \end{tikzcd}$$

that commute up to isomorphism,

(iii) and whose 2-cells $(k, h) \Rightarrow (k', h')$ consist of a pair of 2-cells $\beta\colon k \Rightarrow k'$ and $\gamma\colon h \Rightarrow h'$ between geometric morphisms for which the 2-diagram

$$\begin{tikzcd}[row sep=huge] \mathcal{F} \arrow[r, bend left, "k"{name=U}] \arrow[r, bend right, "k'"'{name=D}] \arrow[d, "f"'] \arrow[from=U, to=D, Rightarrow, "\beta"] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{E} \arrow[r, bend left, "h"{name=U2}] \arrow[r, bend right, "h'"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "\gamma"] & \mathcal{H} \end{tikzcd}$$

commutes (where the empty 2-cells $g \circ k \simeq h \circ f$ and $g \circ k' \simeq h' \circ f$ represent the distinguished isomorphisms), i.e.

$$g * \beta \simeq \gamma * f.$$

Much of the necessary work regarding relative topos theory has been developed in [26], [24] and [8]. Chapter I both recalls this background theory, and extends the previous literature in the necessary directions for our intended, logical applications.

What is the benefit of relative topos theory? By fixing a base topos $\mathcal{E}$ for $\mathcal{F}$, we have presented $\mathcal{F}$ *internally* to $\mathcal{E}$ (see [63, Theorem B3.3.4]). In particular, $\mathcal{F}$ is the topos of *internal sheaves* on an *internal site*. Many internal constructions inside topoi, and in particular internal sites, can be of interest outside of topos theory.

Examples 0.5. For example, given a monoid M, any internal notion of the topos $\mathbf{B}M$ comes equipped with an M-action and internal constructions are naturally M-equivariant.

(i) In particular, the internal language of the topos $\mathbf{B}\mathbb{N}$, for the monoid $(\mathbb{N}, 0, +)$, is exploited in [116] for the study of difference algebra.

(ii) In the topos $\mathbf{B}G$, for a discrete group G, an internal (pointfree) space $\mathbb{X}$ of $\mathbf{B}G$ corresponds to an action of G on a space X (see, for instance, [63, Example C2.5.8(d)] or Section II.3.2). Moreover, the cohomology theory of the topos of *internal sheaves* $\mathbf{Sh}_{\mathbf{B}G}(\mathbb{X})$ coincides with Borel's equivariant cohomology for the group action $G \times X \to X$ (see [45] and [110]).

For some applications of relative topos theory, the literature on *internal sites* (see, for instance, [32] or [63]) is already sufficient. However, as explained in [26], the more flexible notion of *relative sites* can often be preferable as a formalism.

A cylindrical Diaconescu's equivalence. We present a 'cylindrical' variant of the relative 'Diaconescu's equivalence' (so-named for [32]) found in [24, Theorem 3.3] and [8, Theorem 3.6]. Our cylindrical version will play the same role as the standard Diaconescu's equivalence in establishing a classifying topos theory for doctrines.

Internal locale theory

Internal locales and geometric theories. In Chapter II, we pursue a systematic study of one, ubiquitous kind of relative sites: *internal locales*. As observed in [63, Theorem D3.2.5] (and generalised in [99], [22, §7.1] and Section III.4), an internal locale of the *object classifier* $\mathbf{Sets}^{\mathbf{FinSets}}$ may be identified with a single-sorted geometric theory. Thus, to study the algebraic structure of (single-sorted) geometric theories, it suffices to study the internal locale theory of $\mathbf{Sets}^{\mathbf{FinSets}}$ (as performed in Chapter III).

Therefore, it is important to proceed with a well-developed theory of internal locales. For applications, it is especially important to have concrete methods of *externalising* properties of internal locales, as these will be of tangible interest to the practising mathematician. External treatments of internal locale theory appear in [68], [63, §C1.6] and [24].

Pointwise properties of internal locales. We will show that

(i) surjections of internal locales,

(ii) embeddings of internal sublocales,

(iii) and the co-frame operations on the co-frame of internal sublocales

can all be computed *'pointwise'*.

Classifying topoi via doctrines

Classifying topoi for doctrines. Having developed the necessary background material in Chapter I and Chapter II, in Chapter III we develop a theory of classifying topoi for Lawverian doctrines. As exposited in Section III.1, any first order system of deduction, satisfying the weakest of requirements, admits a representation by a doctrine, making them the perfect formalism for our philosophical aims.

Doctrines vs. syntactic categories. A textbook account of classifying topos theory, as can be found in [22], [63], [79], constructs the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a regular, coherent, or geometric theory $\mathbb{T}$ via the *syntactic site*. However, as discussed in Section III.3.2, the fact that a *syntactic category* can be constructed for the theory $\mathbb{T}$ presupposes that it exists in a fragment of logic with at least the expressive power of regular logic.

We demonstrate that, when the necessary underlying structure is present, it is equivalent to represent theories using either doctrines or syntactic categories, in as much as they have equivalent classifying topoi.

The geometric completion of a doctrine

Significant interest has been shown in *syntactic* completions of doctrines, such as the existential and universal completions [119], [121] or the canonical extension [30].

The geometric completion. In Chapter IV, we provide another logical completion: the *geometric completion*. This sends a doctrine to a *geometric doctrine*, a member of a class of doctrines with the expressive power of geometric logic.

Unlike other completions of doctrines considered in the literature, the geometric completion takes a Grothendieck topology as a second argument. As a result, the geometric completion is not only *universal*, but also *semantically invariant* and *idempotent*.

Our development yields a general framework for generating completions of doctrines for sub-fragments of geometric logic. In this fashion, we recover Trotta's existential completion (see Proposition IV.31), as well as identifying the coherent completion of a doctrine. We are also able to relate completions of doctrines to completions of categories, such as the regular completion (see [27]), via the syntactic category construction studied in Section III.3.

Sheaves on a groupoid

Chapter V begins our study of the groupoidal representation of predicate theories. Every localic or topological groupoid comes equipped with a natural notion of a *topos of equivariant sheaves*. Chapter V establishes the pertinent properties of topoi of sheaves on a topological groupoid that will facilitate our subsequent study of representing groupoids. Section V.2 includes a comparison between the representation of topoi by topological groupoids and localic groupoids.

A localic representing groupoid

In the landmark paper [68], Joyal and Tierney famously proved that every topos is equivalent to a topos of equivariant sheaves on an open localic groupoid, giving sense to the statement that every topos is a 'space whose points can possess non-trivial isomorphisms'.

Because the localic representation of classifying topoi contrasts with their topological representation studied in Chapter VII, we give in Chapter VI an explicit description of a representing localic groupoid for the classifying topos of a geometric theory and sketch its providence via the methods of [68]. As expressed in [63, Remark C5.2.8(c)], the argument via geometric theories is the most natural way to witness the Joyal-Tierney representation result. The localic groupoid we give is directly comparable to the representing topological groupoids studied in [5], [36], [37] and, with a slight modification (see Remark VI.23), also the representing topological groupoids studied in [17].

Topological representing groupoids

Topological representation. In [17], Butz and Moerdijk give the topological parallel to the Joyal-Tierney result [68] by showing that every topos with enough points is equivalent to the topos of sheaves on an open topological groupoid. Equivalently, this expresses that every geometric theory whose set-based models are conservative has a representing groupoid of models. In [5], [36], [37], an explicitly logical description of a representing open topological groupoid is given for the classifying topos of a geometric theory with enough points. Special cases of representing groupoids are considered in [21] and [11].

The classification result. Our contribution in Chapter VII is to characterise the open topological groupoids that represent a given geometric theory. Intuitively, this characterises which groupoids of models 'have enough information' to recover the theory.

We will observe that it is not merely enough for the underlying models of a groupoid to be conservative. Instead, we must also impose a model-theoretic condition, *elimination of parameters*, on the groupoid. The representation of geometric theories using doctrines, as set out in Chapter III, is used to simplify our calculations. The results on internal locale morphisms from Chapter II are also essential to our proof.

Our classification result recovers the representations of topoi by open topological groupoids considered in the literature. We are also able to demonstrate, using the classification result, that every open topological groupoid is *Morita equivalent* to its *étale completion*, giving a topological parallel to the same result for localic groupoids found in [92, §7] (see also Remark VI.6).

Representing groupoids for doctrines. Finally, the language of doctrines, used in Part A, is married to the language of representing groupoids from Part B when in Section VII.7 we translate our classification of the representing groupoids of a geometric theory, across the geometric completion from Chapter IV, to deduce a classification of the representing groupoids of a doctrine, thus fulfilling the intended purpose of the geometric completion: to produce one study of semantics within the familiar syntax of geometric logic.

Weak equivalences of groupoids

Moerdijk's equivalence. We continue our study of the groupoid representation of topoi in Chapter VIII. In [92, §7], Moerdijk demonstrates that the category **Topos** is equivalent to a subcategory of localic groupoids localised by a right calculus of fractions (see [40, §1]).

A topological parallel. We aim to give the topological parallel, where topoi are replaced by topoi with enough points, and localic groupoids are replaced by topological groupoids. Informally, this expresses that, just as topoi can be thought of as a 'space whose points have isomorphisms', geometric morphisms can be thought of as 'continuous maps that preserve isomorphisms of points'.

We demonstrate that, unlike the Moerdijk result [92], in the topological setting we cannot obtain our desired equivalence by taking a *right* calculus of fractions on any subcategory of topological groupoids. A *left* (bi)calculus of fractions must instead be employed.

Logical motivation. As explicated in Section VI.1, a geometric morphism

$$f \colon \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}'}$$

between classifying topoi (with enough points) is identical to a (pseudo-natural) functor

$$F_{\mathcal{F}} \colon \mathbb{T}'\text{-mod}(\mathcal{F}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{F})$$

between model categories, or informally: instructions on how to transform a $\mathbb{T}'$-model into a $\mathbb{T}$-model. Our biequivalence thus expresses that the geometric morphism f is determined by the action of $F_{\mathbf{Sets}}$ restricted to a representing groupoid of models for $\mathbb{T}'$, lending credence to the slogan that representing groupoids are those groupoids of models that 'have enough information' to recover the theory.

Moreover, from the biequivalence we are able to deduce a characterisation of Morita equivalence for topological groupoids, and thus a characterisation of Morita equivalence for theories in terms of their representing groupoids.

Part A

Doctrinal Representations

Chapter I

Relative topos theory

What is relative topos theory? There are many equivalent ways to define what a (Grothendieck) topos is.

(i) A topos $\mathcal{E}$ is a category satisfying the *Giraud axioms* (see [79, Appendix]), including the requirement that $\mathcal{E}$ has a *small* set of generators.

(ii) A topos is a category that is equivalent to the category of sheaves on some *site* (in **Sets**, i.a. a small site).

(iii) A topos $\mathcal{E}$ is an *elementary topos* with a *bounded* geometric morphism $\mathcal{E} \to \mathbf{Sets}$ (see [63, Definition A2.1.1, Definition B3.1.7]).

Thus, broadly conceived, (Grothendieck) topos theory is the study of topoi *over* the topos **Sets**.

The focus on **Sets** is not strictly necessary. Given any elementary topos $\mathcal{E}$, a bounded geometric morphism between elementary topoi $f : \mathcal{F} \to \mathcal{E}$ with codomain $\mathcal{E}$ yields an *internal site* $(\mathbb{C}, \mathcal{J})$ of $\mathcal{E}$ (see [63, Theorem B3.3.4]). Even when $\mathcal{F}$ and $\mathcal{E}$ are both Grothendieck topoi, in which case they are both presented by some pair of sites in **Sets**, it may still be valuable to fix a certain geometric morphism $f : \mathcal{F} \to \mathcal{E}$ or *base topos* for $\mathcal{F}$. This is because the choice of geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is akin to a particular 'perspective' on $\mathcal{F}$, relative to $\mathcal{E}$. Relative topos theory embraces this relative perspective – the fundamental objects of study being the geometric morphisms between topoi.

A cylindrical Diaconescu's equivalence. The purpose of this chapter is to exposit a relative site theory, in the sense of [26, §8], suitable for our applications to doctrine theory. Recall that *Diaconescu's equivalence* establishes an equivalence between the geometric morphisms $f : \mathcal{F} \to \mathbf{Sh}(C, J)$ whose codomain is the sheaf topos $\mathbf{Sh}(C, J)$ with certain functors $C \to \mathcal{F}$, the *flat* functors. A relative version of Diaconescu's equivalence appears in [24, Theorem 3.3] and [8, Theorem 3.6], work inspired by a particular case tackled in [42]. In Theorem I.21, we present a 'cylindrical' variant that emphasises a change of base, which will facilitate our definition of the classifying topos of a doctrine in Chapter III.

Relative vs. internal site theory. There are two approaches found in the literature that generalise site theory to a relative setting. In addition to relative sites, as introduced in [26], there is also the internal site theoretic approach, which translates standard site theory into the internal language of a topos (see, for instance, [32]). As also discussed in [26], on the whole relative sites present a more attractive formalism for the study of relative topoi.

(i) Relative sites are a slightly weaker notion than internal sites, and thus more sites can be compared, although every relative site is *Morita equivalent* to an internal site. In this regard, there are two chief benefits to relative sites.
   - Firstly, a relative site may be *large*. The benefit of this can be likened to how the *canonical site* $(\mathcal{E}, J_{\text{can}})$ of a topos $\mathcal{E}$ is not small, and therefore not strictly internal to **Sets**, but it is still a useful site to consider.
   - Secondly, a relative site may be *pseudo-functorial*, and thus higher categorical notions can be considered using relative site theory.

(ii) Being explicit by definition, when working with those relative sites that are also internal sites, there is no need to translate between internal and external notions. This is especially useful for applications outside of topos theory.

Overview. To prefigure the subsequent development: a relative site consists of a *comorphism of sites* that is also a *Street fibration*. The first half of this chapter consists of recalling the necessary background material to make sense of this definition, before the latter half turns to its use in relative topos theory. The chapter is divided as follows.

(A) We first recall in Section I.1 the notions of a comorphism and a morphism of sites.

(B) In Section I.2, we recall the theory of fibrations and their relation to fibred categories. We recall the theory of Grothendieck fibrations and Street fibrations separately, although we will not differentiate between the two.

(C) Having recalled enough background material, in Section I.3 we recall the definition of relative sites and define their morphisms. We show that a morphism of relative sites induces a morphism of relative topoi.

(D) The final addition, Section I.4, is devoted to a cylindrical version of the relative Diaconescu's equivalence found in [24] and [8]. We also prove some consequences of this result, including an extension of the notion of *subcanonical topology* to the relative setting.

I.1 Morphisms and comorphisms of sites

Morphisms and comorphisms of sites constitute two methods of generating geometric morphisms from their generating data, i.e. sites, and can therefore be compared with defining homomorphisms on free algebras by functions on their generators. Indeed, we will observe in Section II.4 that morphisms of sites generalise the practice of defining frame homomorphisms in terms of generators and relations for a frame. Morphisms and comorphisms of sites were originally introduced, under different names, in [3]. For a modern treatment, the reader is directed to [79, §VII] and [23].

Comorphisms of sites. Comorphisms of sites are functors of the underlying categories of sites that induce geometric morphisms covariantly.

Definition I.1 (Definition 2.1, Exposé III [3])**.** Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites. A *comorphism of sites*

$$F\colon (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$$

is a functor $F\colon \mathcal{C} \to \mathcal{D}$ with the *cover lifting property* – for each object c of $\mathcal{C}$ and K-covering sieve S on $F(c)$, there exists a J-covering sieve R on c such that $F(R) \subseteq S$.

A comorphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism

$$C_F\colon \mathbf{Sh}(\mathcal{C}, J) \longrightarrow \mathbf{Sh}(\mathcal{D}, K)$$

(see [3, §III.2] or [79, Theorem VII.10.5]) for which the inverse image C_F^* is given by $\mathbf{a}_J(- \circ F)$. The composite of two comorphisms of sites F and G is still a comorphism of sites whose induced geometric morphism is the composite $C_{F\circ G} = C_F \circ C_G$. Moreover, any natural transformation

$$\begin{tikzcd} (\mathcal{C}, J) \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\gamma"] & (\mathcal{D}, K) \end{tikzcd}$$

between comorphisms of sites induces a natural transformation $\mathbf{a}_J(- \circ F) \Rightarrow \mathbf{a}_J(- \circ G)$ and thus a 2-cell of geometric morphisms $C_F \Rightarrow C_G$.

Thus, taking the geometric morphism induced by a comorphism of sites is naturally bifunctorial. Let **ComorphSites** denote the bicategory whose objects are sites, whose 1-cells are comorphisms of sites and whose 2-cells are natural transformations between comorphisms of sites. By above, there is a bifunctor **ComorphSites** $\to$ **Topos** that sends a site to its topos of sheaves, and a comorphism of sites F to its induced geometric morphism C_F.

The Giraud topology. Recall from [3, §3.1, Exposé III] that, given a functor $F\colon \mathcal{D} \to \mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$, there is a unique finest topology on $\mathcal{D}$ making F a comorphism of sites. In [3], the name *'topologie induite'* was used, but the topology was subsequently dubbed the *Giraud topology* in [26] due to its pioneering use in [42].

Definition I.2 (Definition 3.1, Exposé III [3], cf. §2 [42])**.** Let $(\mathcal{C}, J)$ be a site, and let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor. The *Giraud topology* J_F on $\mathcal{D}$ is the Grothendieck topology on $\mathcal{D}$ defined by the following universal property. For any other Grothendieck topology K on $\mathcal{D}$, the following are equivalent:

(i) firstly, $J_F \subseteq K$;

(ii) the functor F defines a comorphism of sites

$$(\mathcal{D}, K) \xrightarrow{F} (\mathcal{C}, J);$$

(iii) for each J-covering sieve S on $F(d) \in \mathcal{C}$, the sieve

$$\left\{ d' \xrightarrow{g} d \in \mathcal{D} \,\middle|\, F(g) \in S \right\}$$

is K-covering.

Morphisms of sites. Morphisms of sites, on the other hand, are functors of the underlying categories of sites that induce geometric morphisms contravariantly.

Definition I.3 (Definition 3.2 [23])**.** Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites. A *morphism of sites*

$$F\colon (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$$

is a functor $F\colon \mathcal{C} \to \mathcal{D}$ satisfying the following conditions.

(i) If S is a J-covering sieve on $c \in \mathcal{C}$, then $F(S)$ is a K-covering family of morphisms on $F(c)$.

(ii) Every object d of $\mathcal{D}$ admits a K-covering sieve $\{ d_i \to d \mid i \in I \}$ such that each object d_i, for $i \in I$, has a morphism $d_i \to F(c_i)$ to the image of some $c_i \in \mathcal{C}$.

(iii) For any pair of objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms

$$d \xrightarrow{g_1} F(c_1), \qquad d \xrightarrow{g_2} F(c_2)$$

of $\mathcal{D}$, there exists a K-covering family

$$\left\{ d_i \xrightarrow{h_i} d \,\middle|\, i \in I \right\}$$

of morphisms in $\mathcal{D}$, a pair of families

$$\left\{ c_i \xrightarrow{f_i^1} c_1 \,\middle|\, i \in I \right\}, \left\{ c_i \xrightarrow{f_i^2} c_2 \,\middle|\, i \in I \right\}$$

of morphisms in $\mathcal{C}$, and, for each $i \in I$, a morphism $d_i \xrightarrow{k_i} F(c_i')$ such that the squares

$$\begin{tikzcd} d_i \arrow[r, "h_i"] \arrow[d, "k_i"'] & d \arrow[d, "g_1"] \\ F(c_i) \arrow[r, "F(f_i^1)"] & F(c_1), \end{tikzcd} \qquad \begin{tikzcd} d_i \arrow[r, "h_i"] \arrow[d, "k_i"'] & d \arrow[d, "g_2"] \\ F(c_i) \arrow[r, "F(f_i^2)"] & F(c_2) \end{tikzcd}$$

commute.

(iv) For any pair of parallel arrows

$$\begin{tikzcd} c' \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & c \end{tikzcd}$$

of $\mathcal{C}$, and any arrow $d \xrightarrow{h} F(c')$ of $\mathcal{D}$ such that $F(f_1) \circ g = F(f_2) \circ g$, there exists a K-covering family

$$\left\{ d_i \xrightarrow{h_i} d \,\middle|\, i \in I \right\}$$

of morphisms of $\mathcal{D}$, a family of morphisms

$$\left\{ c_i \xrightarrow{e_i} c' \,\middle|\, i \in I \right\}$$

of $\mathcal{C}$ such that $f_1 \circ e_i = f_2 \circ e_i$ for all $i \in I$, and, for each $i \in I$, a morphism $d_i \xrightarrow{k_i} F(c_i)$ such that the square

$$\begin{tikzcd} d_i \arrow[r, "h_i"] \arrow[d, "k_i"'] & d \arrow[d, "g"] \\ F(c_i) \arrow[r, "F(e_i)"] & F(c') \end{tikzcd}$$

commutes for each $i \in I$.

Remark I.4. In Definition I.3, conditions (ii) to (iv) express that a functor preserves finite limits *relatively*, including those finite limits that do not appear in $\mathcal{C}$ (cf. the discussion in [71]). Condition (ii) expresses that the terminal object is preserved, (iii) products, and (iv) equalizers. If $\mathcal{C}$ and $\mathcal{D}$ are both cartesian categories, then a functor $F\colon \mathcal{C} \to \mathcal{D}$ that preserves finite limits satisfies conditions (ii) to (iv). The converse is also true K is a *subcanonical* topology (see [107, Corollary 4.14]).

A morphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism

$$\mathbf{Sh}(F)\colon \mathbf{Sh}(\mathcal{D}, K) \longrightarrow \mathbf{Sh}(\mathcal{C}, J)$$

for which the direct image $\mathbf{Sh}(F)_*$ sends a sheaf $P\colon \mathcal{D}^{op} \to \mathbf{Sets}$ of $\mathbf{Sh}(\mathcal{D}, K)$ to $P \circ F^{op}$ (see [79, Theorem VII.10.2]). Morphisms of sites were originally defined in [3, Definition 1.1, Exposé III] as the hypothesis in the following result:

Proposition I.5 (§3.2 [23])**.** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a morphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$ if and only if there exists a geometric morphism*

$$f\colon \mathbf{Sh}(\mathcal{D}, K) \longrightarrow \mathbf{Sh}(\mathcal{C}, J)$$

such that the square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\ell_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\ell_{\mathcal{D}}"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "f^*"] & \mathbf{Sh}(\mathcal{D}, K) \end{tikzcd}$$

commutes (here, $\ell_{\mathcal{C}}$ denotes the canonical functor *of the site $(\mathcal{C}, J)$, i.e. the composite $\mathbf{a}_J \circ \text{よ}_{\mathcal{C}}$ of the Yoneda embedding followed by sheafification). If so, the geometric morphism f is unique up to unique isomorphism.*

It follows that the composite of two morphisms of sites is still a morphism of sites and that $\mathbf{Sh}(F \circ G) = \mathbf{Sh}(G) \circ \mathbf{Sh}(F)$ for any two composable morphisms of sites F and G. Similarly, a natural transformation

$$\begin{tikzcd} (\mathcal{C}, J) \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D, above}] & (\mathcal{D}, K) \arrow[Rightarrow, from=U, to=D, "\gamma"] \end{tikzcd}$$

between morphisms of sites evidently yields a natural transformation

$$\mathbf{Sh}(F)_* = - \circ F^{\mathrm{op}} \Longrightarrow - \circ G^{\mathrm{op}} = \mathbf{Sh}(G)_*$$

and therefore a 2-cell of geometric morphisms $\mathbf{Sh}(G) \Rightarrow \mathbf{Sh}(F)$ (see also [63, Remark C2.3.5]).

Thus, just as with comorphisms of sites, taking the geometric morphism induced by a morphism of sites is naturally bifunctorial. Let **MorphSites** denote the bicategory whose objects are sites, whose 1-cells are morphisms of sites and whose 2-cells are natural transformations between morphisms of sites. By above, there exists a bifunctor $\mathbf{MorphSites}^{\mathrm{op}} \to \mathbf{Topos}$ that sends a site to its topos of sheaves and a morphism of sites to its induced geometric morphism.

Dense morphisms of sites. Many properties of geometric morphisms can be computed at the level of morphisms and comorphisms of sites, as is demonstrated in [23]. Of particular interest is when a morphism of sites induces an equivalence of topoi. Sufficient conditions are described in [75, §2]. Necessary and sufficient conditions are given in [23, Proposition 5.5 & Theorem 5.7], however we won't need the extra generality.

Definition I.6 (§2 [75])**.** A *dense morphism of sites*

$$F \colon (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$$

is a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that:

(i) S is a J-covering family in $\mathcal{C}$ if and only if $F(S)$ is K-covering in $\mathcal{D}$,

(ii) for every object d of $\mathcal{D}$, there exists a K-covering family of morphisms $F(c_i) \to d$,

(iii) for every pair of objects c_1, c_2 of $\mathcal{C}$ and an arrow $F(c_1) \xrightarrow{g} F(c_2)$ in $\mathcal{D}$, there is a J-covering family of arrows $c_i' \xrightarrow{f_i} c_1$ and a family of arrows $c_i' \xrightarrow{k_i} c_2$ such that $g \circ F(f_i) = F(k_i)$,

(iv) for any pair of arrows

$$\begin{tikzcd} c_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & c_2 \end{tikzcd}$$

in $\mathcal{C}$ such that $F(f_1) = F(f_2)$, there exists a J-covering family of arrows

$$\left\{ c_i' \xrightarrow{k_i} c_1 \,\middle|\, i \in I \right\}$$

such that $f_1 \circ k_i = f_2 \circ k_i$ for all $i \in I$.

By [107, Theorem 11.2], each dense morphism of sites is a morphism of sites. The induced geometric morphism is an equivalence of topoi.

The comparison lemma, as originally formulated for (full) subcategories in [3, Theorem 4.1, Exposé III], can be recovered via the special case of a dense morphism of sites whose underlying functor is the inclusion of a subcategory.

Definition I.7. A subcategory $C \subseteq \mathcal{D}$ of a site $(\mathcal{D}, K)$ is *dense* if

(i) for every $d \in \mathcal{D}$, there is a covering family $S \in K(d)$ generated by morphisms whose domains are in C,

(ii) for every arrow $c \xrightarrow{g} d \in \mathcal{D}$, there is a covering family $S \in J(c)$ generated by morphisms $b \xrightarrow{f} c$ such that $g \circ f$ is in C.

Lemma I.8 (The Comparison Lemma)**.** *Let $(\mathcal{D}, K)$ be a site and let C be a dense subcategory. There is an equivalence of topoi $\mathbf{Sh}(\mathcal{D}, K) \simeq \mathbf{Sh}(C, K|_C)$.*

I.2 Fibrations

In this section, we recall the theory of fibrations. Just as the datum of a presheaf $P\colon C^{\mathrm{op}} \to \mathbf{Sets}$ can be collected into a single category using the *category of elements* construction, so too can the datum of a functor $P\colon C^{\mathrm{op}} \to \mathbf{Cat}$, or a pseudo-functor $P\colon C^{\mathrm{op}} \to \mathfrak{CAT}$, be concentrated into a single category: the *Grothendieck construction* $C \rtimes P$ (see [44, Eposé IV]). The category $C \rtimes P$ comes equipped with a canonical projection $\pi_P\colon C \rtimes P \to C$, and fibrations characterise those functors of this form. We would expect the theory of fibrations to appear in the study of internal sites since, by [63, Corollary D1.2.14], an internal category of a presheaf topos $\mathbf{Sets}^{C^{\mathrm{op}}}$ is simply a functor $P\colon C^{\mathrm{op}} \to \mathbf{Cat}$.

I.2.1 Grothendieck fibrations

Although we will eventually consider the more general notion of *Street fibration*, for ease of development we first recall the theory of (Grothendieck) fibrations. Recall from [63, Definition B1.3.4] that a fibration $A\colon C \to \mathcal{E}$ is a functor such that, for each object c of C and an arrow $e \xrightarrow{f} A(c)$, there exists a *cartesian lifting* $d \xrightarrow{g} c$ of f, that is an arrow of C such that $A(g) = f$ and, for any arrows $d' \xrightarrow{g'} c$ of C and $A(d') \xrightarrow{k} A(d)$ of $\mathcal{E}$ for which the triangle

$$\begin{tikzcd} A(d') \arrow[r, "k"] \arrow[dr, "A(g')"'] & A(d) \arrow[d, "A(g)"] \\ & A(c) \end{tikzcd}$$

commutes, there exists a unique arrow $d' \xrightarrow{k'} d$ of C such that the triangle

$$\begin{tikzcd} d' \arrow[r, "k'"] \arrow[dr, "g'"'] & d \arrow[d, "g"] \\ & c \end{tikzcd}$$

commutes and $A(k') = k$ (note that we are using the terminology 'cartesian arrow' where Johnstone uses 'prone'). Recall also that, given a pair of fibrations $A\colon C \to \mathcal{E}$ and $B\colon \mathcal{D} \to \mathcal{F}$, a *morphism of the fibrations* $A \to B$ consists of a pair of functors

$F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{E} \to \mathcal{F}$ such that the square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "A"'] & \mathcal{D} \arrow[d, "B"] \\ \mathcal{E} \arrow[r, "G"] & \mathcal{F} \end{tikzcd}$$

commutes and, if $d \xrightarrow{g} c \in \mathcal{C}$ is cartesian, so too is $F(d) \xrightarrow{F(g)} F(c)$.

Cloven fibrations and fibred categories. Recall that a *cleavage* for the fibration A is a choice of cartesian lifting for each arrow $e \xrightarrow{f} A(e)$.

Any *(strictly) fibred category*, i.e. a functor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, yields a fibration via the *Grothendieck construction*. We denote by $\mathcal{C} \rtimes P$ the category

(i) whose objects are pairs (c, x) where c is an object of $\mathcal{C}$ and x is an object of $P(c)$,

(ii) and an arrow $(c, x) \xrightarrow{(f,g)} (d, y)$ is a pair consisting of an arrow $c \xrightarrow{f} d$ of $\mathcal{C}$ and an arrow $x \xrightarrow{g} P(f)(y)$ of $P(c)$.

The projection functor $\pi_P\colon \mathcal{C} \rtimes P \to \mathcal{C}$, which acts by

$$\begin{aligned} (c, x) &\mapsto c, \\ (c, x) \xrightarrow{(f,g)} (d, y) &\mapsto c \xrightarrow{f} d, \end{aligned}$$

is a fibration. The cartesian lifting of an arrow $e \xrightarrow{f} \pi_P(c, x)$ in $\mathcal{C}$ can be taken as the arrow $(e, P(f)(x)) \xrightarrow{(f, \mathrm{id}_{P(f)(x)})} (c, x)$, yielding a cleavage for the fibration $\pi_P\colon \mathcal{C} \rtimes P \to \mathcal{C}$.

Example I.9. The Grothendieck construction of a presheaf $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$, viewed as a strictly fibred category, coincides with the *category of elements* of P.

Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ be a pair of strictly fibred categories. A *morphism of strictly fibred categories*, by which we mean a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha\colon P \Rightarrow Q \circ F^{\mathrm{op}}$, also yields a morphism of fibrations. We will denote by $F \rtimes \alpha$ the functor $F \rtimes \alpha\colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$ that acts by

$$\begin{aligned} (c, x) &\mapsto (F(c), \alpha_c(x)), \\ (c, x) \xrightarrow{(f,g)} (d, y) &\mapsto (F(c), \alpha_c(x)) \xrightarrow{(F(f), \alpha_c(g))} (F(d), \alpha_d(y)). \end{aligned}$$

The square

$$\begin{tikzcd} \mathcal{C} \rtimes P \arrow[r, "F \rtimes \alpha"] \arrow[d, "\pi_P"'] & \mathcal{D} \rtimes Q \arrow[d, "\pi_Q"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

commutes, and $F \rtimes \alpha$ sends cartesian arrows to cartesian arrows. Therefore, the pair $(F, F \rtimes \alpha)$ is a morphism of fibrations $\pi_P \to \pi_Q$.

Definition I.10 (Definition 7.1, Exposé IV [44])**.** Fibrations of the form $\pi_P\colon \mathcal{C} \rtimes P \to \mathcal{C}$ are known as *cloven* fibrations.

Proposition I.11 (Theorem 1.3.5 [63]). *If we assume the axiom of choice, every fibration is cloven.*

Therefore, we will often elect to work in the notionally and conceptually convenient framework of (strictly) fibred categories rather than fibrations.

I.2.2 Street fibrations

For the majority of this thesis, we will be working with functors valued in *skeletal categories*, in particular **PoSet**, where every isomorphism is an equality, and therefore the above formalism suffices for our applications. However, for the more general development of the theory of relative sites, fibrations will not be enough. Namely, fibrations break the so-called 'principle of equivalence' – that constructions in category theory should only be defined up to equivalence and not equality. Instead, we must work with a variation of the above theory of fibrations.

The necessary generalisations to the flavour of fibrations studied above were introduced by Street in [113] and developed further in [114]. Just as cloven fibrations correspond to strictly fibred categories, cloven Street fibrations correspond to fibred categories in the bifunctorial sense (see [10]).

Definitions I.12 (Definition 4.1 [10], Definition 2.8 [114]). (i) By a *fibred category* we mean a *pseudo-functor* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{CAT}$, essentially a functor where we have relaxed the condition that P preserves identities and compositions of arrows; P now only needs to preserve these up to equivalence, i.e. P consists of the data:

a) a category $P(c)$ for each $c \in \mathcal{C}$,

b) a functor $P(f)\colon P(c) \to P(d)$ for each $d \xrightarrow{f} c \in \mathcal{C}$,

c) a distinguished natural isomorphism $P_{\mathrm{id}_c}\colon \mathrm{id}_{P(c)} \xRightarrow{\sim} P(\mathrm{id}_c)$ for each $c \in \mathcal{C}$,

d) and a distinguished natural isomorphism

$$P_{f\circ g}\colon P(g)\circ P(f) \xRightarrow{\sim} P(f\circ g)$$

for each pair $e \xrightarrow{g} d$, $d \xrightarrow{f} c \in \mathcal{C}$,

satisfying the coherence axioms:

e) for each arrow $d \xrightarrow{f} c \in \mathcal{C}$,

$$\begin{tikzcd} P(c) \arrow[r, "P(\mathrm{id}_c)"'] \arrow[r, bend left=60, "\mathrm{id}_{P(c)}"] \arrow[rr, bend right=50, "P(f)"'] \arrow[r, phantom, bend left=30, "\wr\Downarrow P_{\mathrm{id}_c}"] & P(c) \arrow[r, "P(f)"] \arrow[d, phantom, "\wr\Downarrow P_{\mathrm{id}_c\circ f}"] & P(d) \arrow[r, phantom, "="] & P(c) \arrow[r, "P(f)"] \arrow[rr, bend right=50, "P(f)"'] & P(d) \arrow[r, "P(\mathrm{id}_d)"'] \arrow[r, bend left=60, "\mathrm{id}_{P(d)}"] \arrow[r, phantom, bend left=30, "\wr\Downarrow P_{\mathrm{id}_d}"] \arrow[d, phantom, "P_{f\circ \mathrm{id}_d}\Downarrow\wr"] & P(d) \\ & {} & & & {} & \end{tikzcd}$$

f) and for each triple of arrows $e' \xrightarrow{h} e$, $e \xrightarrow{g} d$, $d \xrightarrow{f} c \in C$,

$$\begin{tikzcd}
& P(d) \arrow[dr, "P(g)"] \arrow[d, phantom, "\overset{\sim}{\Downarrow} P_{f\circ g}"] & & \\
P(c) \arrow[ur, bend left, "P(f)"] \arrow[r, "P(f\circ g)"'] \arrow[rr, bend right=60, "P(f\circ g\circ h)"'] & P(e) \arrow[r, "P(h)"] \arrow[d, phantom, "\overset{\sim}{\Downarrow} P_{(f\circ g)\circ h}"] & P(e') \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
& & P(e) \arrow[dr, "P(h)"] \arrow[d, phantom, "\overset{\sim}{\Downarrow} P_{g\circ h}"] & \\
P(c) \arrow[r, "P(f)"] \arrow[rrr, bend right=50, "P(f\circ g\circ h)"'] & P(d) \arrow[ur, bend left, "P(g)"] \arrow[rr, "P(g\circ h)"'] \arrow[d, phantom, "P_{f\circ(g\circ h)} \overset{\sim}{\Downarrow}"] & & P(e') \\
& {} & &
\end{tikzcd}$$

(ii) A *Street fibration* is a functor $A\colon C \to \mathcal{E}$ such that for each object $c \in C$ and an arrow $e \xrightarrow{f} A(c)$, there exists a (weak) cartesian lifting $d \xrightarrow{g} c$ of f, by which we mean that there exists a distinguished isomorphism $h\colon e \xrightarrow{\sim} P(d)$ such that $P(g) \circ h = f$ and that g is cartesian in the same sense as before – i.e. for any arrows $d' \xrightarrow{g'} c \in C$ and $A(d') \xrightarrow{k} A(d) \in C$ for which the triangle

$$\begin{tikzcd}
A(d') \arrow[r, "k"] \arrow[dr, "A(g')"'] & A(d) \arrow[d, "A(g)"] \\
& A(c)
\end{tikzcd}$$

commutes, there exists a unique arrow $d' \xrightarrow{k'} d$ of C such that the triangle

$$\begin{tikzcd}
d' \arrow[r, "k'"] \arrow[dr, "g'"'] & d \arrow[d, "g"] \\
& c
\end{tikzcd}$$

commutes and $A(k') = k$.

Proposition I.13 (Corollary 3.8 [114])**.** *For each fibred category*

$$P\colon C^{\mathrm{op}} \longrightarrow \mathfrak{CAT},$$

the Grothendieck construction $C \rtimes P$ yields a Street fibration $\pi_P\colon C \rtimes P \to C$.

Example I.14 (The canonical fibration of a geometric morphism)**.** We give some examples of pseudo-functors that are not functors, and hence Street fibrations that are not Grothendieck fibrations, culminating in what will become for us a recurring example of a Street fibration: the *canonical fibration of a geometric morphism*.

(i) For a category C with all pullbacks, we obtain a pseudo-functor

$$C/(-)\colon C^{\mathrm{op}} \longrightarrow \mathfrak{CAT}$$

by sending each object $c \in C$ to the slice category C/c and, for each arrow $d \xrightarrow{f} c \in C$, the functor C/f acts by sending $e \xrightarrow{g} c \in C/c$ to the pullback

$$\begin{tikzcd}
e' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, "g"] \\
d \arrow[r, "f"] & c.
\end{tikzcd}$$

Since pullbacks are only defined up to unique isomorphism, in general $C/(-)$ is a pseudo-functor not a functor. The corresponding Street fibration coincides with the functor

$$\mathrm{tgt}\colon C^2 \longrightarrow C$$

whose domain is the arrow category C^2 of C and which acts on objects by sending an arrow to its target.

(ii) We can perform a relative version of the above construction. For a category C with all pullbacks and and any functor $F\colon \mathcal{D} \to C$, we obtain, analogously to above, a pseudo-functor $C/F\colon \mathcal{D}^{\mathrm{op}} \to \mathfrak{CAT}$ by sending each object $d \in \mathcal{D}$ to the slice category $C/F(d)$ and each arrow $d' \xrightarrow{f} d \in C$ to the functor $C/F(f)$. The corresponding category $\mathcal{D} \rtimes C/F$ coincides with the *comma category*, usually denoted by $(\mathrm{id}_C \downarrow F)$.

In particular, a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ yields pseudo-functor

$$\mathcal{F}/f^*\colon \mathcal{E}^{\mathrm{op}} \longrightarrow \mathfrak{CAT}$$

and hence a Street fibration $\mathcal{E} \rtimes \mathcal{F}/f^* \to \mathcal{E}$, the *canonical fibration of a geometric morphism*.

Morphisms of Street fibrations. We should also eliminate the use of equality in our definition of a morphism of fibrations. We therefore define a *morphism of Street fibrations* $A\colon C \to \mathcal{E}$ and $B\colon \mathcal{D} \to \mathcal{F}$ to be a pair of functors $F\colon C \to \mathcal{D}$ and $G\colon \mathcal{E} \to \mathcal{F}$ such that F sends cartesian arrows to cartesian arrows and the square

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "A"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "B"] \\ \mathcal{E} \arrow[r, "G"'] & \mathcal{F} \end{tikzcd}$$

commutes up to natural isomorphism. Given two fibred categories $P\colon C^{\mathrm{op}} \to \mathfrak{CAT}$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathfrak{CAT}$, a *morphism of fibred categories* consists of a functor $F\colon C \to \mathcal{D}$ and *pseudo-natural transformation* $\alpha\colon P \to Q \circ F^{\mathrm{op}}$, that is a functor $\alpha_c\colon P(c) \to QF(c)$ for each $c \in C$, and a natural isomorphism $\alpha_f\colon QF(f) \circ \alpha_c \xRightarrow{\sim} \alpha_d\colon P(f)$, for each arrow $d \xrightarrow{f} c \in C$, satisfying the coherence conditions

$$\begin{tikzcd} P(c) \arrow[r, bend left, "\mathrm{id}_{P(c)}"] \arrow[r, bend right, "P(\mathrm{id}_c)"'] \arrow[r, phantom, "\wr\Downarrow P_{\mathrm{id}_c}"] & P(c) \arrow[r, "\alpha_c"] & Q(c) \end{tikzcd} = \begin{tikzcd}[row sep=large] & QF(c) \arrow[r, bend left, "\mathrm{id}_{QF(c)}"] \arrow[r, bend right, "Q(\mathrm{id}_{F(c)})"'] \arrow[r, phantom, "\wr\Downarrow Q_{\mathrm{id}_{F(c)}}"] \arrow[d, Rightarrow, "\wr\ \alpha_{\mathrm{id}_c}"] & QF(c) \\ P(c) \arrow[ur, "\alpha_c"] \arrow[r, "P(\mathrm{id}_c)"'] & P(c) \arrow[ur, "\alpha_c"'] & \end{tikzcd}$$

and, for each pair $e \xrightarrow{g} d, d \xrightarrow{f} c \in C$,

$$
\begin{tikzcd}
 & & QF(d) \arrow[dr, "QF(g)"] \arrow[d, phantom, "\wr\Downarrow Q_{F(f)\circ F(g)}"] & \\
P(c) \arrow[r, "\alpha_c"] \arrow[drr, bend right, "P(f\circ g)"'] & QF(c) \arrow[ur, "QF(f)"] \arrow[rr, "QF(f\circ g)"'] & {} \arrow[d, phantom, "\wr\Downarrow\alpha_{f\circ g}"] & QF(e) \\
 & & P(e) \arrow[ur, "\alpha_e"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & QF(c) \arrow[drr, "QF(f)"] \arrow[d, phantom, "\wr\Downarrow\alpha_f"] & & & \\
P(c) \arrow[ur, "\alpha_c"] \arrow[r, "P(f)"] \arrow[drrr, bend right, "P(f\circ g)"'] & P(d) \arrow[r, "\alpha_d"] \arrow[d, phantom, "P_{f\circ g}\Downarrow\wr"] \arrow[drr, "P(g)"] & {} & QF(d) \arrow[r, "QF(g)"] \arrow[d, phantom, "\wr\Downarrow\alpha_g"] & QF(e). \\
 & {} & & P(e) \arrow[ur, "\alpha_e"'] &
\end{tikzcd}
$$

As before, the pair (F, α) yields a morphism of Street fibrations $(F, F \rtimes \alpha)\colon \pi_P \to \pi_Q$ (see [26, §2.2]).

Since fibrations in the sense of Grothendieck and the sense of Street share the same pertinent properties, we will not bother differentiating the two notions. Indeed, a cloven Street fibration is equivalent to a Grothendieck fibration (see [26, Proposition 2.2.5]).

I.3 Relative sites

We have now recalled enough background material to recall the definition of a relative site.

Definition I.15 (Definition 8.2.1 [26])**.** Let (C, J) be a site. A *relative site* over (C, J) consists of

(i) a fibration $A\colon \mathcal{D} \to C$,

(ii) and a topology K on C such that K contains the Giraud topology J_A (see Definition I.2).

Equivalently, a relative site is a fibred category $P\colon C^{\mathrm{op}} \to \mathfrak{CAT}$ and a topology K on $C \rtimes P$ such that the fibration yields a comorphism of sites $\pi_P\colon (C \rtimes P, K) \to (C, J)$. Thus, (modulo some size requirements discussed below) every relative site defines a relative topos

$$C_{\pi_P}\colon \mathbf{Sh}(C \rtimes P, K) \longrightarrow \mathbf{Sh}(C, J).$$

Remark I.16. Although our fibred category $P\colon C^{\mathrm{op}} \to \mathfrak{CAT}$ is allowed to take values in large categories, there are some size requirements we must impose. Firstly, we require that, for each $c \in C$, the category $P(c)$ is locally small. Secondly, we require that the site $(C \rtimes P, K)$ has a *small set of generators*, i.e. a small set of objects $\{ (c_i, x_i) \mid i \in I \} \subseteq C \rtimes P$ such that any other object $(d, y) \in C \rtimes P$ admits a K-covering by arrows whose domains are taken among the set $\{ (c_i, x_i) \mid i \in I \}$. All of the examples of fibred categories with topologies on $C \rtimes P$ one considers in practice satisfy these size conditions.

These conditions are necessary since we desire the two facts to be true of a relative site:

(i) the category of sheaves $\mathbf{Sh}(C \rtimes P, K)$ is a (Grothendieck) topos (in particular, $\mathbf{Sh}(C \rtimes P, K)$ has a small set of generators),

(ii) and secondly, there is a Yoneda functor $よ_{C \rtimes P} : C \rtimes P \to \mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}}$ and hence also a *canonical functor* $\ell_{C \rtimes P} : C \rtimes P \to \mathbf{Sh}(C \rtimes P, K)$.

The notion of a relative site generalises the notion of an internal site (see [63, §C2]). An internal category of a presheaf topos $\mathbf{Sets}^{C^{\mathrm{op}}}$, i.e. a functor $P : C^{\mathrm{op}} \to \mathbf{Cat}$, can be assigned a topos of *internal presheaves*, which is given by the presheaf topos $\mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}}$ (see [63, Lemma C2.5.3]). An internal Grothendieck topology can also be introduced, and it is shown in [63, Proposition C2.5.4] that the topos of *internal sheaves* on an internal site of $\mathbf{Sh}(C, J)$ is of the form $\mathbf{Sh}(C \rtimes P, K)$ for some Grothendieck topology K on $C \rtimes P$ containing the Giraud topology. Thus, when a relative site is also an internal site, our development coincides with the internal treatment.

Example I.17 (§8.2.2 [26], The canonical relative site of a geometric morphism)**.** We return to the example of the canonical relative fibration $\mathcal{E} \rtimes \mathcal{F}/f^*$ of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ from Example I.14 and describe a topology $\tilde{J}_{\mathrm{can}}$ on the category $\mathcal{E} \rtimes \mathcal{F}/f^* \to \mathcal{E}$ for which there is an equivalence

$$\mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \simeq \mathcal{F},$$

from which $(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ deserves the title: the *canonical relative site of a geometric morphism*.

Recall that objects of $\mathcal{E} \rtimes \mathcal{F}/f^*$ are pairs

$$\left(E\,,\, F \xrightarrow{g} f^*E\right)$$

where $E \in \mathcal{E}$ and $F \xrightarrow{g} f^*E \in \mathcal{F}$, and an arrow

$$\left(E\,,\, F \xrightarrow{g} f^*E\right) \xrightarrow{(h,h')} \left(E'\,,\, F' \xrightarrow{g'} f^*E'\right)$$

consists of a pair of arrows $E \xrightarrow{h} E' \in \mathcal{E}$ and $F \xrightarrow{h'} F' \in \mathcal{F}$ such that the square

$$\begin{array}{ccc} F & \xrightarrow{h'} & F' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ E & \xrightarrow{F(h)} & E' \end{array}$$

in $\mathcal{F}$ commutes. There are two evident projections: firstly, the ever-present fibration

$$\pi_{\mathcal{F}/f^*} : \mathcal{E} \rtimes \mathcal{F}/f^* \longrightarrow \mathcal{E}$$

that sends $\left(E\,,\, F \xrightarrow{g} f^*E\right)$ to E (for increased symmetry of notation, we will denote this functor by $\pi_{\mathcal{E}}$). Secondly, there is the projection $\pi_{\mathcal{F}} : \mathcal{E} \rtimes \mathcal{F}/f^* \to \mathcal{F}$,

$$\left(E\,,\, F \xrightarrow{g} f^*E\right) \mapsto F,$$
$$\left(E\,,\, F \xrightarrow{g} f^*E\right) \xrightarrow{(h,h')} \left(E'\,,\, F' \xrightarrow{g} f^*E'\right) \mapsto F \xrightarrow{h'} F'.$$

We denote by $\tilde{J}_{\mathrm{can}}$ the Grothendieck topology on $\mathcal{E} \rtimes \mathcal{F}/f^*$ whose covering sieves are precisely those sieves that are sent by $\pi_{\mathcal{F}}$ to jointly epimorphic families, i.e. a family of arrows

$$\left\{ \left(E_i\,,\, F_i \xrightarrow{g} f^*E_i \right) \xrightarrow{(h_i, h_i')} \left(E\,,\, F \xrightarrow{g} f^*E \right) \,\middle|\, i \in I \right\}$$

in $\mathcal{E} \rtimes \mathcal{F}/f^*$ is $\tilde{J}_{\mathrm{can}}$-covering if and only if

$$\left\{ F_i \xrightarrow{h_i'} F \,\middle|\, i \in I \right\}$$

is a jointly epimorphic family in $\mathcal{F}$. As $f^*\colon \mathcal{E} \to \mathcal{F}$ preserves jointly epimorphic families, the projection

$$\pi_{\mathcal{E}}\colon (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \longrightarrow (\mathcal{E}, J_{\mathrm{can}})$$

has the cover lifting property and so is a comorphism of sites. Hence, $(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ is a relative site over the canonical site $(\mathcal{E}, J_{\mathrm{can}})$ for $\mathcal{E}$.

In fact, we can say more. By applying [23, Theorem 3.16], the functor $\pi_{\mathcal{F}}$ is both a dense morphism of sites and a comorphism of sites

$$\pi_{\mathcal{F}}\colon (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \longrightarrow (\mathcal{F}, J_{\mathrm{can}}),$$

from which we deduce an equivalence of topoi $\mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \simeq \mathcal{F}$, and moreover that there is an isomorphism of geometric morphisms

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d, phantom, "\wr\|"] \arrow[dr, phantom, "\cong"] & \mathcal{E} \arrow[d, phantom, "\wr\|"] \\
\mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \arrow[r, "C_{\pi_{\mathcal{E}}}"'] & \mathcal{E}.
\end{tikzcd}$$

Thus, every geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is represented by its canonical relative site $\pi_{\mathcal{E}}\colon (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \to (\mathcal{E}, J_{\mathrm{can}})$.

I.3.1 Morphisms of relative sites

We complete this section by describing morphisms of relative sites. Our theory is a natural extension to that developed in [24] so as to include change of base.

Definition I.18. A *morphism of relative sites*

$$(F, G)\colon \left[(\mathcal{C}, J) \xrightarrow{A} (\mathcal{E}, L) \right] \longrightarrow \left[(\mathcal{D}, K) \xrightarrow{B} (\mathcal{F}, M) \right]$$

consists of a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{E} \to \mathcal{F}$ such that

(i) the pair (F, G) constitutes a morphism of fibrations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "A"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "B"] \\
\mathcal{E} \arrow[r, "G"'] & \mathcal{F},
\end{tikzcd}$$

(ii) and both $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$ and $G\colon (\mathcal{E}, L) \to (\mathcal{F}, M)$ are morphisms of sites.

Later in Proposition I.28 we will observe that morphisms of relative sites admit a simpler description in special cases. Let

$$\begin{tikzcd} (\mathcal{C}, J) \arrow[r, "F"] \arrow[d, "A"'] \arrow[dr, phantom, "\cong"] & (\mathcal{D}, K) \arrow[d, "B"] \\ (\mathcal{E}, L) \arrow[r, "G"] & (\mathcal{F}, M), \end{tikzcd}$$

be a morphism of relative sites. The constituent functors A, B, F and G induce a square of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[d, "C_A"'] & \mathbf{Sh}(\mathcal{D}, K) \arrow[l, "\mathbf{Sh}(F)"'] \arrow[d, "C_B"] \\ \mathbf{Sh}(\mathcal{E}, L) & \mathbf{Sh}(\mathcal{F}, M). \arrow[l, "\mathbf{Sh}(G)"'] \end{tikzcd} \tag{I.i}$$

However, *a priori* there is no reason for this square to commute (up to isomorphism) and thus define a morphism of relative topoi. We will show that the specific conditions made on a morphism of relative sites, namely that the functors A, B, F and G also constitute a morphism of fibrations, suffice to demonstrate that the square (I.i) does indeed commute.

Lemma I.19. *Let $A\colon (\mathcal{C}, J) \to (\mathcal{E}, L)$ and $B\colon (\mathcal{D}, K) \to (\mathcal{F}, M)$ both be relative sites and let*

$$(F, G)\colon \left[(\mathcal{C}, J) \xrightarrow{A} (\mathcal{E}, L)\right] \longrightarrow \left[(\mathcal{D}, K) \xrightarrow{B} (\mathcal{F}, M)\right]$$

be a morphism of relative sites. Then the induced square of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[d, "C_A"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{D}, K) \arrow[l, "\mathbf{Sh}(F)"'] \arrow[d, "C_B"] \\ \mathbf{Sh}(\mathcal{E}, L) & \mathbf{Sh}(\mathcal{F}, M) \arrow[l, "\mathbf{Sh}(G)"'] \end{tikzcd}$$

commutes up to isomorphism.

Proof. The overarching method of the proof is to turn the morphisms of sites F and G into comorphisms of sites, and then appeal to the bifunctoriality of sending a comorphism of sites to its induced geometric morphism. We are able to turn morphisms of sites into comorphisms of sites by [23, Theorem 3.16]. For the morphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$, there are functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "i_F"] & (1_{\mathcal{D}} \downarrow F) \arrow[l, shift left, "\pi_{\mathcal{C}}"] \arrow[r, "\pi_{\mathcal{D}}"] & \mathcal{D} \end{tikzcd}$$

where

(i) $(1_{\mathcal{D}} \downarrow F)$ denotes the comma category

a) whose objects are pairs
$$\left(c, d \xrightarrow{a} F(c)\right)$$
of an object $c \in \mathcal{C}$ and an arrow $d \to F(c)$ in $\mathcal{D}$,

b) and whose arrows are pairs
$$(c', d' \xrightarrow{a'} F(c')) \xrightarrow{(g,h)} (c, d \xrightarrow{a} F(c))$$
of arrows $c' \xrightarrow{g} c \in \mathcal{C}$ and $d' \xrightarrow{h} d'$ for which the square
$$\begin{tikzcd} d' \arrow[r, "h"] \arrow[d, "a'"'] & d \arrow[d, "a"] \\ F(c') \arrow[r, "F(g)"] & F(c) \end{tikzcd}$$
commutes;

(ii) $\pi_{\mathcal{C}} \colon (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ and $\pi_{\mathcal{D}} \colon (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ are the respective projection functors
$$\pi_{\mathcal{C}} \colon (c, d \xrightarrow{a} F(c)) \mapsto c,$$
$$\pi_{\mathcal{D}} \colon (c, d \xrightarrow{a} F(c)) \mapsto d;$$

(iii) $i_F \colon \mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ is the functor that sends $c \in \mathcal{C}$ to
$$\left(c, F(c) \xrightarrow{\mathrm{id}_{F(c)}} F(c)\right) \in (1_{\mathcal{D}} \downarrow F).$$

Moreover, when the category $(1_{\mathcal{D}} \downarrow F)$ is endowed with the Grothendieck topology $\tilde{K}$, whose covering sieves are precisely those that are sent by $\pi_{\mathcal{D}}$ to K-covering sieves, we have that

(i) $\pi_{\mathcal{C}} \colon ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$ is a comorphism of sites,

(ii) $i_F \colon (\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ is a morphism of sites,

(iii) $\pi_{\mathcal{D}} \colon ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism and comorphism of sites and induces an equivalence of topoi
$$\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \simeq \mathbf{Sh}(\mathcal{D}, K).$$

We also have that $\mathbf{Sh}(F) = C_{\pi_{\mathcal{C}}} \circ \mathbf{Sh}(\pi_{\mathcal{D}})$, and $C_{\pi_{\mathcal{D}}}$ is an inverse to $\mathbf{Sh}(\pi_{\mathcal{D}})$. Similarly, there are functors
$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "i_G"] & (1_{\mathcal{F}} \downarrow G) \arrow[l, shift left, "\pi_{\mathcal{E}}"] \arrow[r, "\pi_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd}$$
with analogous properties, in particular $\mathbf{Sh}(G) = C_{\pi_{\mathcal{E}}} \circ \mathbf{Sh}(\pi_{\mathcal{F}})$ and $C_{\pi_{\mathcal{F}}}$ is an inverse for $\mathbf{Sh}(\pi_{\mathcal{F}})$.

We construct a comorphism of sites $H \colon ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to ((1_{\mathcal{F}} \downarrow G), \tilde{M})$ such that the diagram
$$\begin{tikzcd} \mathcal{C} \arrow[d, "A"'] \arrow[dr, phantom, "\cong"] & (1_{\mathcal{D}} \downarrow F) \arrow[l, "\pi_{\mathcal{C}}"'] \arrow[r, "\pi_{\mathcal{D}}"] \arrow[d, "H"] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "B"] \\ \mathcal{E} & (1_{\mathcal{F}} \downarrow G) \arrow[l, "\pi_{\mathcal{E}}"'] \arrow[r, "\pi_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd} \tag{I.ii}$$

commutes up to isomorphism. The functor H sends an object $\left(c, d \xrightarrow{a} F(c)\right)$ to

$$\left(A(c), B(d) \xrightarrow{B(a)} B(F(c)) \cong G(A(c))\right),$$

where we have used that the square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "A"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "B"] \\ \mathcal{E} \arrow[r, "G"] & \mathcal{F}, \end{tikzcd} \tag{I.iii}$$

commutes up to isomorphism. Similarly, H is defined to send an arrow

$$\left(c', d' \xrightarrow{a'} F(c')\right) \xrightarrow{(g,h)} \left(c, d \xrightarrow{a} F(c)\right)$$

to $(A(h), B(g))$. The functor H clearly makes the diagram (I.ii) commute up to isomorphism.

It remains to show that H has the cover lifting property. Let

$$S = \left\{ \left(e_i, f_i \xrightarrow{b} G(e_i)\right) \xrightarrow{(g_i,h_i)} \left(A(c), B(d) \xrightarrow{B(a)} B(F(c)) \cong G(A(c))\right) \middle| i \in I \right\}$$

be a $\tilde{M}$-covering sieve, i.e.

$$\pi_{\mathcal{F}}(S) = \left\{ f_i \xrightarrow{h_i} B(d) \middle| i \in I \right\}$$

is M-covering. As A is a fibration, there exists, for each $i \in I$, a cartesian lifting of $e_i \xrightarrow{g_i} A(c) \in \mathcal{E}$ to an arrow $c' \xrightarrow{g'} c \in \mathcal{C}$. Since the square (I.iii) is also a morphism of fibrations, $F(c') \xrightarrow{F(g')} F(c) \in \mathcal{D}$ is cartesian too. Now we apply the fact that B has the cover lifting property to deduce the existence of a K-covering sieve R on d such that $B(R) \subseteq \pi_{\mathcal{F}}(S)$, i.e. for each $d' \xrightarrow{k} d$ in R, there exists an $i \in I$ such that $B(k)$ factors as

$$\begin{tikzcd} B(d') \arrow[r, dashed] \arrow[rr, bend left, "B(k)"] & f_i \arrow[r, "h_i"] \arrow[d, "b"'] & B(d) \arrow[d, "B(a)"] \\ & B(F(c')) \arrow[r, "B(F(g'))"] & B(F(c)) \cong G(A(c)). \end{tikzcd}$$

As $F(g')$ is cartesian, there is a unique arrow $d' \xrightarrow{\gamma} F(c') \in \mathcal{D}$ making the square

$$\begin{tikzcd} d' \arrow[r, "k"] \arrow[d, "\gamma"'] & d \arrow[d, "a"] \\ F(c') \arrow[r, "F(g')"] & F(c) \end{tikzcd}$$

commute. Hence, as R is a K-covering sieve,

$$\left\{ \left(c', d \xrightarrow{\gamma} F(c')\right) \xrightarrow{(k,g')} \left(c, d \xrightarrow{a} F(c)\right) \middle| k \in R \right\}$$

is a $\tilde{K}$-covering lifting of S, whence H is a comorphism of sites

$$H \colon ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to ((1_{\mathcal{F}} \downarrow G), \tilde{M})$$

as desired.

By the commutation of (I.ii) up to isomorphism, we deduce that the induced diagram of geometric morphisms

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}, J) \arrow[d, "C_A"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \arrow[l, "C_{\pi_{\mathcal{C}}}"'] \arrow[r, "C_{\pi_{\mathcal{D}}}"] \arrow[d, "C_H"] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{D}, K) \arrow[d, "C_B"] \\
\mathbf{Sh}(\mathcal{E}, L) & \mathbf{Sh}((1_{\mathcal{F}} \downarrow G), \tilde{M}) \arrow[l, "C_{\pi_{\mathcal{E}}}"] \arrow[r, "C_{\pi_{\mathcal{F}}}"] & \mathbf{Sh}(\mathcal{F}, M)
\end{tikzcd}$$

commutes up to isomorphism too. Thereby, we conclude that

$$\begin{aligned}
C_A \circ \mathbf{Sh}(F) &= C_A \circ C_{\pi_{\mathcal{C}}} \circ \mathbf{Sh}(\pi_{\mathcal{D}}), \\
&\simeq C_{\pi_{\mathcal{E}}} \circ C_H \circ \mathbf{Sh}(\pi_{\mathcal{D}}), \\
&= C_{\pi_{\mathcal{E}}} \circ \mathbf{Sh}(\pi_{\mathcal{F}}) \circ C_{\pi_{\mathcal{F}}} \circ C_H \circ \mathbf{Sh}(\pi_{\mathcal{D}}), \\
&\simeq \mathbf{Sh}(G) \circ C_B \circ C_{\pi_{\mathcal{D}}} \circ \mathbf{Sh}(\pi_{\mathcal{D}}), \\
&= \mathbf{Sh}(G) \circ C_B
\end{aligned}$$

as required. □

I.4 A cylindrical Diaconescu's equivalence

In this final section, we present a cylindrical version of the relative Diaconescu's equivalence and some corollaries of our statement. Recall that Diaconescu's equivalence states that, for each site $(\mathcal{C}, J)$ and each topos $\mathcal{E}$, there is an equivalence of categories

$$\mathbf{Topos}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq J\text{-}\mathbf{Flat}(\mathcal{C}, \mathcal{E}). \tag{I.iv}$$

Unravelling definitions, the latter category is precisely

$$\mathbf{MorphSites}\left((\mathcal{C}, J), (\mathcal{E}, J_{\text{can}})\right).$$

One direction of the equivalence (I.iv) sends a J-flat functor $F \colon \mathcal{C} \to \mathcal{E}$ to the geometric morphism

$$\mathbf{Sh}(F) \colon \mathcal{E} \simeq \mathbf{Sh}(\mathcal{E}, J_{\text{can}}) \longrightarrow \mathbf{Sh}(\mathcal{C}, J)$$

induced by F as a morphism of sites $F \colon (\mathcal{C}, J) \to (\mathcal{E}, J_{\text{can}})$. In the other direction, a geometric morphism $f \colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ is sent to the J-flat functor $f^* \circ \ell_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{E}$. Diaconescu's equivalence is essential to the textbook development of classifying topos theory. The same is true for our relative exposition.

A relative version of Diaconescu's equivalence appears in [24, Theorem 3.3] and [8, Theorem 3.6], generalising the work of Giraud [42]. We present a 'cylindrical' variant to the equivalence found in [8], [24] that includes a change of base suitable for our later applications. We provide a self-contained account here, but the functors witnessing the equivalence in Theorem I.21 could also be constructed by appealing to [8, Theorem 3.6] (see Remark I.22).

For notational convenience in this section, when there is no confusion, we will write a^{-1} for the functor $P(a)\colon P(d) \to P(c)$, where $P\colon \mathbf{C}^{\mathrm{op}} \to \mathfrak{CAT}$ is a fibred category and $c \xrightarrow{a} d$ is an arrow of $\mathbf{C}$.

Lemma I.20. *Let $P\colon \mathbf{C}^{\mathrm{op}} \to \mathfrak{CAT}$ be a fibred category. For each object (d,y) of $\mathbf{C} \rtimes P$, there exists a canonical choice of a natural transformation*

$$\text{ら}_{(d,y)}\colon \text{よ}_{\mathbf{C}\rtimes P}(d,y) \longrightarrow \text{よ}_{\mathbf{C}}(d) \circ \pi_P.$$

such that

(i) for each arrow $z \xrightarrow{u} y$ of $P(d)$, the triangle

$$\begin{tikzcd} \text{よ}_{\mathbf{C}\rtimes P}(d,z) \arrow[rr, "(\mathrm{id}_c{,}u)\circ-"] \arrow[dr, "\text{ら}_{(d,z)}"'] & & \text{よ}_{\mathbf{C}\rtimes P}(d,y) \arrow[dl, "\text{ら}_{(d,y)}"] \\ & \text{よ}_{\mathbf{C}}(d) \circ \pi_P & \end{tikzcd}$$

commutes,

(ii) and for each arrow $c \xrightarrow{a} d$ of $\mathbf{C}$, the square

$$\begin{tikzcd} \text{よ}_{\mathbf{C}\rtimes P}(c,a^{-1}y) \arrow[r, "(a{,}\mathrm{id}_{a^{-1}y})\circ-"] \arrow[d, "\text{ら}_{(c,a^{-1}y)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \text{よ}_{\mathbf{C}\rtimes P}(d,y) \arrow[d, "\text{ら}_{(d,y)}"] \\ \text{よ}_{\mathbf{C}}(c) \circ \pi_P \arrow[r, "a\circ-"] & \text{よ}_{\mathbf{C}}(d) \circ \pi_P \end{tikzcd}$$

is a pullback in $\mathbf{Sets}^{(\mathbf{C}\rtimes P)^{\mathrm{op}}}$.

Proof. For each object (e,x), the map

$$\text{ら}^{(e,x)}_{(d,y)}\colon \text{よ}_{\mathbf{C}\rtimes P}(d,y)(e,x) \to \text{よ}_{\mathbf{C}}(d)(e)$$

that sends an arrow $(e,x) \xrightarrow{(f,v)} (d,y)$ to $e \xrightarrow{f} d$ is evidently the component of a natural transformation. Immediately, we see that the triangle

$$\begin{tikzcd} \text{よ}_{\mathbf{C}\rtimes P}(d,z) \arrow[rr, "(\mathrm{id}_c{,}u)\circ-"] \arrow[dr, "\text{ら}_{(d,z)}"'] & & \text{よ}_{\mathbf{C}\rtimes P}(d,y) \arrow[dl, "\text{ら}_{(d,y)}"] \\ & \text{よ}_{\mathbf{C}}(d) \circ \pi_P & \end{tikzcd}$$

commutes for each arrow $z \xrightarrow{u} y \in P(d)$.

Recalling that pullbacks in $\mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}$ are computed pointwise, for (ii) is suffices to show that, for each $(e,x) \in C \rtimes P$, the square

$$\begin{tikzcd}
よ_{C\rtimes P}(c, a^{-1}y)(e,x) \arrow[r, "(a,\mathrm{id}_{a^{-1}y})\circ -"] \arrow[d, "ら^{(e,x)}_{(c,a^{-1}y)}"'] & よ_{C\rtimes P}(d,y)(e,x) \arrow[d, "ら^{(e,x)}_{(d,y)}"] \\
よ_C(c)(e) \arrow[r, "a\circ -"] & よ_C(d)(e)
\end{tikzcd}$$

is a pullback in **Sets**. This is given by the evident isomorphism

$$\begin{aligned}
よ_{C\rtimes P}(c, a^{-1}y)(e,x) &= \left\{ (e,x) \xrightarrow{(f,u)} (c, a^{-1}y) \;\middle|\; \begin{array}{c} e \xrightarrow{f} c \in C, \\ x \xrightarrow{u} a^{-1}y \in P(e) \end{array} \right\}, \\
&\cong \left\{ \left(e \xrightarrow{f} c, (e,x) \xrightarrow{(g,u)} (d,y)\right) \;\middle|\; \begin{array}{c} e \xrightarrow{f} c \in C, \\ x \xrightarrow{u} a^{-1}y \in P(e), \\ e \xrightarrow{g} d \in C, \\ g = a\circ f \end{array} \right\}, \\
&= よ_C(c)(e) \times_{よ_C(d)(e)} よ_{C\rtimes P}(d,y)(e,x).
\end{aligned}$$

□

Theorem I.21 (The cylindrical Diaconescu's equivalence)**.** *Given a relative site over (C,J), i.e. a pseudo-functor*

$$P\colon C^{\mathrm{op}} \to \mathfrak{CAT}$$

and a Grothendieck topology K on $C \rtimes P$ that contains the Giraud topology, there is an equivalence of categories

$$\mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(C\rtimes P, K) \\ \Big\downarrow f, & \Big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(C,J) \end{array}\right) \simeq \mathbf{RelMorph}\left(\begin{array}{cc} (C\rtimes P, K) & (\mathcal{E}\rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \Big\downarrow \pi_P \quad , & \Big\downarrow \pi_{\mathcal{E}} \\ (C,J) & (\mathcal{E}, J_{\mathrm{can}}) \end{array}\right),$$

where

$$\mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(C\rtimes P, K) \\ \Big\downarrow f, & \Big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(C,J) \end{array}\right)$$

denotes the category

(i) whose objects are squares of geometric morphisms

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(C\rtimes P, K) \arrow[d, "C_{\pi_P}"] \\
\mathcal{E} \arrow[r, "h"] & \mathbf{Sh}(C,J)
\end{tikzcd}$$

that commute up to isomorphism,

(ii) and whose arrows $(g, h) \to (g', h')$ consist of a pair of 2-cells $\beta\colon g \Rightarrow g'$ and $\gamma\colon h \Rightarrow h'$ for which the cylindrical 2-diagram

$$
\begin{tikzcd}
\mathcal{F} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[d, "f"'] & & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\
\mathcal{E} \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & \mathbf{Sh}(\mathcal{C}, J)
\end{tikzcd}
$$

commutes (where the empty 2-cells represent the distinguished isomorphisms), i.e.

$$C_{\pi_P} * \beta \simeq \gamma * f,$$

and

$$
\mathbf{RelMorph}\left(
\begin{array}{c}
(\mathcal{C} \rtimes P, K) \\
\downarrow \pi_P \\
(\mathcal{C}, J)
\end{array}
,
\begin{array}{c}
(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\text{can}}) \\
\downarrow \pi_{\mathcal{E}} \\
(\mathcal{E}, J_{\text{can}})
\end{array}
\right)
$$

denotes the category

(i) whose objects are morphisms of relative sites, i.e. morphisms of fibrations

$$
\begin{tikzcd}
\mathcal{C} \rtimes P \arrow[r, "G"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\pi_{\mathcal{E}}"] \\
\mathcal{C} \arrow[r, "H"] & \mathcal{E}
\end{tikzcd}
$$

for which $G\colon (\mathcal{C} \rtimes P, K) \to (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\text{can}})$ and $H\colon (\mathcal{C}, J) \to (\mathcal{E}, J_{\text{can}})$ are both morphisms of sites,

(ii) and whose arrows $(G, H) \xrightarrow{(\beta,\gamma)} (G', H')$ consist of a pair of natural transformations $\beta\colon G \Rightarrow G'$ and $\gamma\colon H \Rightarrow H'$ for which the cylindrical 2-diagram

$$
\begin{tikzcd}
(\mathcal{C} \rtimes P, K) \arrow[rr, bend left, "G"] \arrow[rr, bend right, "G'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[d, "\pi_P"'] & & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\text{can}}) \arrow[d, "\pi_{\mathcal{E}}"] \\
(\mathcal{C}, J) \arrow[rr, bend left, "H"] \arrow[rr, bend right, "H'"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & (\mathcal{E}, J_{\text{can}})
\end{tikzcd}
$$

commutes (i.e. $\pi_{\mathcal{E}} * \beta \simeq \gamma * \pi_P$).

Proof. Given a morphism of fibrations

$$\begin{tikzcd} \mathcal{C} \rtimes P \arrow[r, "G"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\pi_{\mathcal{F}}"] \\ \mathcal{C} \arrow[r, "H"] & \mathcal{E} \end{tikzcd}$$

for which $G\colon (\mathcal{C} \rtimes P, K) \to (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ and $H\colon (\mathcal{C}, J) \to (\mathcal{E}, J_{\mathrm{can}})$ are both morphisms of sites, by Lemma I.19 and Example I.17 there is a diagram of geometric morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[r, phantom, "\simeq"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \arrow[r, "\mathbf{Sh}(G)"] \arrow[d, "C_{\pi_{\mathcal{F}}}"] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\ \mathcal{E} \arrow[r, equal, "\sim"] & \mathbf{Sh}(\mathcal{E}, J_{\mathrm{can}}) \arrow[r, "\mathbf{Sh}(H)"] & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

that commutes up to isomorphism.

This assignment of objects can be extended to a functor

$$\mathbf{RelMorph}\left(\begin{array}{cc} (\mathcal{C} \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P \quad , & \big\downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right) \longrightarrow \mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \big\downarrow f \; , & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array} \right)$$

since, recalling from [63, Remark C2.3.5], any natural transformations $\beta\colon G \Rightarrow G'$ and $\gamma\colon H \Rightarrow H'$ between morphisms of sites induce 2-cells $\mathbf{Sh}(\beta)\colon \mathbf{Sh}(G) \Rightarrow \mathbf{Sh}(G')$ and $\mathbf{Sh}(\gamma)\colon \mathbf{Sh}(H) \Rightarrow \mathbf{Sh}(H')$ on the induced geometric morphisms. It remains to show that these 2-cells satisfy the necessary commutativity condition that, if $\pi_{\mathcal{E}} * \beta \simeq \gamma * \pi_P$, then $C_{\pi_P} * \mathbf{Sh}(\beta) = \mathbf{Sh}(\gamma) * f$. We can essentially perform the same construction as in Lemma I.19 – transforming the morphisms of sites into comorphisms of sites, and natural transformations of morphisms of sites into natural transformations of comorphisms of sites, and then appealing to the bifunctoriality of the comorphism of sites to geometric morphism construction.

We now construct the converse functor

$$\mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \big\downarrow f \; , & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array} \right) \longrightarrow \mathbf{RelMorph}\left(\begin{array}{cc} (\mathcal{C} \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P \quad , & \big\downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right).$$

Given a square of geometric morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\ \mathcal{E} \arrow[r, "h"] & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

that commutes up to isomorphism, by precomposing the inverse image functor h^* with the canonical functor $\ell_C \colon C \to \mathbf{Sh}(C, J)$, as in the standard Diaconescu's equivalence (I.iv), a morphism of sites $H \colon (C, J) \to (\mathcal{E}, J_{\mathrm{can}})$ is obtained. Constructing the complementary morphism of sites

$$G \colon (C \rtimes P, K) \longrightarrow (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$$

is more involved.

We wish to construct a (pseudo-)natural transformation

$$\begin{tikzcd} C^{\mathrm{op}} \arrow[rr, bend left, "P"{name=U}] \arrow[rr, bend right, "\mathcal{F}/f^*H"'{name=D}] & & \mathfrak{CAT}. \arrow[from=U, to=D, Rightarrow, shorten=2mm] \end{tikzcd}$$

That is, we need, for each object $c \in C$, a functor $\mathfrak{g}_c \colon P(c) \to \mathcal{F}/f^*H(c)$ such that, for each arrow $c \xrightarrow{a} d \in C$, the square

$$\begin{tikzcd} P(d) \arrow[rr, "P(a)"] \arrow[d, "\mathfrak{g}_d"'] & {} \arrow[d, phantom, "\cong"] & P(c) \arrow[d, "\mathfrak{g}_c"] \\ \mathcal{F}/f^*H(d) \arrow[rr, "\mathcal{F}/f^*H(a)"] & {} & \mathcal{F}/f^*H(c) \end{tikzcd} \tag{I.v}$$

commutes up to coherent natural isomorphism.

Recall from Lemma I.20 that for each object $y \in P(d)$ there is a natural transformation

$$\text{ら}_{(d,y)} \colon \text{よ}_{C \rtimes P}(d, y) \longrightarrow \text{よ}_C(d) \circ \pi_P,$$

i.e. an arrow of $\mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}}$. By applying $g^*\mathbf{a}_K$ to $\text{ら}_{(d,y)}$, we obtain an arrow

$$g^*\mathbf{a}_K(\text{ら}_{(d,y)}) = \mathfrak{g}_d(y) \colon g^*\mathbf{a}_K \text{よ}_{C \rtimes P}(d, y) \longrightarrow g^*\mathbf{a}_K(\text{よ}_C(d) \circ \pi_P)$$

of $\mathcal{F}$. Recall that $C^*_{\pi_P}$ acts by sending a J-sheaf $F \colon C^{\mathrm{op}} \to \mathbf{Sets}$ to

$$\mathbf{a}_K(F \circ \pi_P) \colon C \rtimes P^{\mathrm{op}} \to \mathbf{Sets}.$$

In particular, $C^*_{\pi_P}(\ell_C(d)) = \mathbf{a}_K(\text{よ}_C(d) \circ \pi_P)$. Therefore,

$$g^*\mathbf{a}_K(\text{よ}_C(d) \circ \pi_P) = g^* C^*_{\pi_P} \ell_C(d) \cong f^* h^* \ell_C(d) = f^*H(d).$$

Thus, $\mathfrak{g}_d(y) \colon g^*\mathbf{a}_K \text{よ}_{C \rtimes P}(d, y) \to g^*\mathbf{a}_K(\text{よ}_C(d) \circ \pi_P)$ is indeed an object of $\mathcal{F}/f^*H(d)$. That this choice of object in $\mathcal{F}/f^*H(d)$ extends to a functor $P(d) \to \mathcal{F}/f^*H(d)$ follows by Lemma I.20(i).

To show that the square (I.v) commutes, we use Lemma I.20(ii). Since

$$\begin{tikzcd} \text{よ}_{C \rtimes P}(c, a^{-1}y) \arrow[r, "(a, \mathrm{id}_{a^{-1}y}) \circ -"] \arrow[d, "\text{ら}_{(c,a^{-1}y)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \text{よ}_{C \rtimes P}(d, y) \arrow[d, "\text{ら}_{(d,y)}"] \\ \text{よ}_C(c) \circ \pi_P \arrow[r, "a \circ -"] & \text{よ}_C(d) \circ \pi_P \end{tikzcd}$$

is a pullback, and $g^*\mathbf{a}_K$ preserves finite limits, the square

$$\begin{tikzcd}
g^*\mathbf{a}_K\,\text{よ}_{C\rtimes P}(c,a^{-1}y) \arrow[rr] \arrow[d, "\mathfrak{g}_c(a^{-1}y)"'] \arrow[drr, phantom, "\lrcorner" very near start] & & g^*\mathbf{a}_K\,\text{よ}_{C\rtimes P}(d,y) \arrow[d, "\mathfrak{g}_d(y)"] \\
f^*H(c) \arrow[rr, "f^*H(a)"] & & f^*H(d)
\end{tikzcd}$$

is a pullback too. Thus, there is a coherent natural isomorphism

$$\mathcal{F}/f^*H(a)(\mathfrak{g}_d(y)) \cong \mathfrak{g}_d(a^{-1}y).$$

We define $G\colon C \rtimes P \to \mathcal{E} \rtimes \mathcal{F}/f^*$ to be the functor $H \rtimes \mathfrak{g}$. Immediately, the pair (G,H) yields a morphism of fibrations

$$\begin{tikzcd}
C \rtimes P \arrow[r, "G"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\pi_{\mathcal{F}}"] \\
C \arrow[r, "H"] & \mathcal{E}.
\end{tikzcd}$$

By the way we have constructed G, the square

$$\begin{tikzcd}
C \rtimes P \arrow[r, "G"] \arrow[d, "\ell_{C\rtimes P}"'] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\ell_{\mathcal{E}\rtimes\mathcal{F}/f^*}"] \\
\mathbf{Sh}(C \rtimes P, K) \arrow[r, "g^*"] & \mathcal{F} \simeq \mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})
\end{tikzcd}$$

commutes, from which we conclude by Proposition I.5 that the functor G is a morphism of sites

$$G\colon (C \rtimes P, K) \longrightarrow (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}).$$

Having constructed the action on objects of the functor

$$\mathbf{Topos}\begin{pmatrix} \mathcal{F} & \mathbf{Sh}(C \rtimes P, K) \\ \big\downarrow f \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(C, J) \end{pmatrix} \longrightarrow \mathbf{RelMorph}\begin{pmatrix} (C \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P & , \qquad \big\downarrow \pi_{\mathcal{E}} \\ (C, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{pmatrix},$$

we now demonstrate that this can be made functorial. Given 2-cells $\beta\colon g \Rightarrow g'$ and $\gamma\colon h \Rightarrow h'$ between geometric morphisms for which the 2-diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[dd, "f"'] & & \mathbf{Sh}(C \rtimes P, K) \arrow[dd, "C_{\pi_P}"] \\
& & \\
\mathcal{E} \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & \mathbf{Sh}(C, J)
\end{tikzcd}$$

commutes, we immediately obtain a natural transformation $\gamma' \colon H \Rightarrow H'$ between the induced morphisms of sites $H, H' \colon (C, J) \rightrightarrows (\mathcal{E}, J_{\text{can}})$ by taking the horizontal composite

$$\begin{tikzcd} C \arrow[r, "\ell_C"] & \mathbf{Sh}(C,J) \arrow[r, bend left, "h^*"{name=U}] \arrow[r, bend right, "h'^*"'{name=D}] & \mathcal{E}. \arrow[from=U, to=D, Rightarrow, "\gamma"] \end{tikzcd}$$

The 2-cells β and γ also yield a natural transformation $\beta' \colon G \Rightarrow G'$ between the induced morphisms of sites $G, G' \colon (C \rtimes P, K) \rightrightarrows (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\text{can}})$. The component

$$\left(H(d), g^*\ell'(d,y) \xrightarrow{g_d(y)} f^*H(d)\right) \xrightarrow{\beta'_{(d,y)}} \left(H'(d), g'^*\ell'(d,y) \xrightarrow{g'_d(y)} f^*H'(d)\right)$$

of β' at $(d, y) \in C \rtimes P$ is given by the pair of morphisms

$$\left(\begin{array}{cc} \begin{tikzcd} H(d) \arrow[d, "\gamma'_d"] \\ H'(d) \end{tikzcd} , & \begin{tikzcd} g^*\ell_{C\rtimes P}(d,y) \arrow[r, "g_d(y)"] \arrow[d, "\beta_{\ell_{C\rtimes P}(d,y)}"'] & f^*H(d) \cong g^*C^*_{\pi_P}(\ell_C(d)) \arrow[d, "\beta_{C^*_{\pi_P}\ell_C(d)}"] \\ g'^*\ell_{C\rtimes P}(y) \arrow[r, "g'_d(y)"] & f^*H'(d) \cong g'^*C^*_{\pi_P}(\ell_C(d)) \end{tikzcd} \end{array}\right).$$

The naturality of β and γ ensures that β' is a natural transformation too.

Finally, it remains to show that the two functors we have constructed define an equivalence of categories. Given a square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(C \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\ \mathcal{E} \arrow[r, "h"] & \mathbf{Sh}(C, J), \end{tikzcd} \tag{I.vi}$$

of geometric morphisms that commutes up to isomorphism, we wish to show that $\mathbf{Sh}(G) \simeq g$ and $\mathbf{Sh}(H) \simeq h$. We use the property from Proposition I.5 that, for a morphism of sites $K \colon (\mathcal{D}, L) \to (\mathcal{D}', L')$, there is a unique geometric morphism $k \colon \mathbf{Sh}(\mathcal{D}', L') \to \mathbf{Sh}(\mathcal{D}, L)$ for which the square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "\ell_{\mathcal{D}}"'] & \mathcal{D}' \arrow[d, "\ell_{\mathcal{D}'}"] \\ \mathbf{Sh}(\mathcal{D}, L) \arrow[r, "k^*"] & \mathbf{Sh}(\mathcal{D}', L') \end{tikzcd}$$

commutes. By this property, or equivalently the standard Diaconescu's equivalence (I.iv), we obtain the latter required equivalence $\mathbf{Sh}(H) \simeq h$.

For the former, we note that the equivalence $\mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\text{can}}) \simeq \mathcal{F}$, being induced by the projection $\pi_{\mathcal{F}} \colon \mathcal{E} \rtimes \mathcal{F}/f^* \to \mathcal{F}$,

$$\left(E, F \xrightarrow{k} f^*E\right) \mapsto F$$

acting as a morphism of sites $\pi_{\mathcal{F}} \colon (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \to (\mathcal{F}, J_{\mathrm{can}})$, identifies a representable $\ell_{\mathcal{E}\rtimes\mathcal{F}/f^*}\left(E, F \xrightarrow{k} f^*E\right) \in \mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ with the object $F \in \mathcal{F}$. Thus, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \rtimes P \arrow[r, "G"] \arrow[d, "\ell_{\mathcal{C}\rtimes P}"'] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\ell_{\mathcal{E}\rtimes\mathcal{F}/f^*}"] \\
\mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[r, "\mathbf{Sh}(G)^*"] \arrow[r, bend right, "g^*"'] & \mathbf{Sh}(\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \simeq \mathcal{F},
\end{tikzcd}$$

and so we obtain the second desired equivalence $\mathbf{Sh}(G) \simeq g$. This is evidently natural, and demonstrates that one composite of our constructed functors, where we begin with a pair of geometric morphisms (g, h) as in (I.vi) and obtain a second pair $(\mathbf{Sh}(G), \mathbf{Sh}(H))$, is naturally isomorphic to the identity. An identical argument demonstrates that the opposite composite is also naturally isomorphic to the identity, thus completing the equivalence. □

Remark I.22. The functor

$$\mathbf{Topos}\begin{pmatrix} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \big\downarrow f \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{pmatrix} \longrightarrow \mathbf{RelMorph}\begin{pmatrix} (\mathcal{C} \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P & , \quad \big\downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{pmatrix}$$

witnessing the equivalence from Theorem I.21 can also be constructed by appealing to [8, Theorem 3.6]. Given an object of

$$\mathbf{Topos}\begin{pmatrix} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \big\downarrow f \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{pmatrix}$$

there is a triangle

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "g"] \arrow[dr, "h \circ f"'] & \cong & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[dl, "C_{\pi_P}"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

to which we can apply [8, Theorem 3.6], yielding the triangle

$$\begin{tikzcd}
(\mathcal{C} \rtimes P, K) \arrow[rr, "\hat{G}"] \arrow[dr, "\pi_P"'] & \cong & (\mathcal{F}/f^*\ell_{\mathcal{C}}, \tilde{J}_{\mathrm{can}}) \arrow[dl] \\
& (\mathcal{C}, J) &
\end{tikzcd}$$

that factors as

$$\begin{tikzcd}
(\mathcal{C} \rtimes P, K) \arrow[r, "G"] \arrow[d, "\pi_P"] & (\mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \arrow[d, "\pi_{\mathcal{E}}"] \\
(\mathcal{C}, J) \arrow[r, "H"] & (\mathcal{E}, J_{\mathrm{can}}),
\end{tikzcd}$$

which gives the action on objects. Given an arrow of

$$\mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \downarrow{\scriptstyle f}, & \downarrow{\scriptstyle C_{\pi_P}} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array}\right)$$

there is a *lax* triangle

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[dr, bend right, "h' \circ f"'] & \arrow[d, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[dl, bend left, "C_{\pi_P}"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

to which we can also apply [8, Theorem 3.6], which yields, after a similar manipulation, the action on arrows.

The relative Diaconescu's equivalence. The specific statements of the relative Diaconescu's equivalence given in [24] and [8], without change of base category, can be recovered by restricting the equivalence in Theorem I.21 to the relevant subcategories, as described below.

Corollary I.23 (Theorem 3.3 [24], Theorem 3.6 [8])**.** *For a topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ and a relative site over $(\mathcal{C}, J)$, there is an equivalence of categories*

$$\mathbf{Topos}/\mathrm{id}_{\mathcal{E}}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \downarrow{\scriptstyle f}, & \downarrow{\scriptstyle C_{\pi_P}} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array}\right) \simeq \mathbf{RelMorph}/\ell_{\mathcal{C}}\left(\begin{array}{cc} (\mathcal{C} \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \downarrow{\scriptstyle \pi_P} \quad , & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ (\mathcal{C}, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{array}\right),$$

where

$$\mathbf{Topos}/\mathrm{id}_{\mathcal{E}}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \downarrow{\scriptstyle f}, & \downarrow{\scriptstyle C_{\pi_P}} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array}\right) \subseteq \mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \downarrow{\scriptstyle f}, & \downarrow{\scriptstyle C_{\pi_P}} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array}\right)$$

is the subcategory

(i) *whose objects are squares of geometric morphisms*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\
\mathcal{E} \arrow[r, equal] & \mathbf{Sh}(\mathcal{C}, J)
\end{tikzcd}$$

that commute up to isomorphism,

(ii) *and whose arrows $g \to g'$ are 2-cells $\beta\colon g \Rightarrow g'$ for which the 2-diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[d, "f"'] & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\
\mathcal{E} \arrow[r, equal] & \mathbf{Sh}(\mathcal{C}, J)
\end{tikzcd}$$

commutes,

and

$$\mathbf{RelMorph}/\ell_{\mathcal{C}} \left(\begin{array}{ccc} (\mathcal{C} \rtimes P, K) & & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \downarrow \pi_P & , & \downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right)$$

denotes the subcategory of

$$\mathbf{RelMorph} \left(\begin{array}{ccc} (\mathcal{C} \rtimes P, K) & & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \\ \downarrow \pi_P & , & \downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right)$$

(i) *whose objects are morphisms of fibrations*

$$\begin{tikzcd}
\mathcal{C} \rtimes P \arrow[r, "G"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \rtimes \mathcal{F}/f^* \arrow[d, "\pi_{\mathcal{E}}"] \\
\mathcal{C} \arrow[r, "\ell_{\mathcal{C}}"] & \mathbf{Sh}(\mathcal{C}, J) \simeq \mathcal{E}
\end{tikzcd}$$

for which $G\colon (\mathcal{C} \rtimes P, K) \to (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ is a morphisms of sites,

(ii) *and whose arrows $G \xrightarrow{\beta} G'$ are natural transformations $\beta\colon G \Rightarrow G'$ for which the 2-diagram*

$$\begin{tikzcd}
(C \rtimes P, K) \arrow[rr, bend left, "G"] \arrow[rr, bend right, "G'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[d, "\pi_P"'] & & (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}}) \arrow[d, "\pi_{\mathcal{E}}"] \\
(C, J) \arrow[rr, "\ell_C"] & & (\mathcal{E}, J_{\mathrm{can}})
\end{tikzcd}$$

commutes.

When $(C \rtimes P, K)$ is an internal site of $\mathbf{Sh}(C, J)$, Corollary I.23 coincides with the internal version of Diaconescu's equivalence.

I.4.1 Fibred preorders

Suppose that, in the statement of Diaconescu's equivalence (I.iv), we assumed C to be a preorder (and therefore $\mathbf{Sh}(C, J)$ is a localic topos – see [57, Theorem 5.37]). Then every flat functor $F\colon C \to \mathcal{E}$ factors through the subcategory of subterminals $\mathrm{Sub}_{\mathcal{E}}(1)$, and so Diaconescu's equivalence (I.iv) becomes

$$\mathbf{Topos}(\mathcal{E}, \mathbf{Sh}(C, J)) \simeq \mathbf{MorphSites}\Big((C, J), (\mathrm{Sub}_{\mathcal{E}}(1), J_{\mathrm{can}})\Big).$$

The relativised version also holds whenever our relative site takes values in **PreOrd**.

Notation I.24. Below, $\mathrm{Sub}_{\mathcal{F}}(f^*-)$ denotes the composite of the opposite of the inverse image functor $f^{*\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{F}^{\mathrm{op}}$ of a geometric morphism with the *subobject doctrine* $\mathrm{Sub}_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{op}} \to \mathbf{PreOrd}$ of the topos $\mathcal{F}$.

Corollary I.25. *Given a site (C, J), a pseudo-functor $P\colon C^{op} \to$ **PreOrd** and a Grothendieck topology K on $C \rtimes P$ that contains the Giraud topology, there is an equivalence of categories*

$$\mathbf{Topos}\left(\begin{array}{c} \mathcal{F} \\ \downarrow f \\ \mathcal{E} \end{array}, \begin{array}{c} \mathbf{Sh}(C \rtimes P, K) \\ \downarrow C_{\pi_P} \\ \mathbf{Sh}(C, J) \end{array}\right) \simeq \mathbf{RelMorph}\left(\begin{array}{c} (C \rtimes P, K) \\ \downarrow \pi_P \\ (C, J) \end{array}, \begin{array}{c} (\mathcal{E} \rtimes \mathrm{Sub}_{\mathcal{F}}(f^*-), \tilde{J}_{\mathrm{can}}) \\ \downarrow \pi_{\mathcal{F}} \\ (\mathcal{E}, J_{\mathrm{can}}) \end{array}\right).$$

Proof. It suffices to show that, for every square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(C \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\
\mathcal{E} \arrow[r, "h"] & \mathbf{Sh}(C, J)
\end{tikzcd}$$

of geometric morphisms that commutes up to isomorphism, and every $(d, y) \in C \rtimes P$, the arrow

$$g^*\mathbf{a}_K\, \text{よ}_{C \rtimes P}(d, y) \xrightarrow{\mathfrak{g}_d(y)} g^*\mathbf{a}_K(\text{よ}_C(d) \circ \pi_P) \cong f^*H(d) \in \mathcal{E} \rtimes \mathcal{F}/f^*H(d)$$

defined in Theorem I.21 is a monomorphism since then the induced morphism of sites $G\colon (C \rtimes P, K) \to (\mathcal{E} \rtimes \mathcal{F}/f^*, \tilde{J}_{\mathrm{can}})$ factors through the subcategory

$$\mathcal{E} \rtimes \mathrm{Sub}_{\mathcal{F}}(f^*-) \subseteq \mathcal{E} \rtimes \mathcal{F}/f^*.$$

As P takes values in **PreOrd**, the natural transformation

$$\text{ち}_{(d,y)}\colon\ \text{よ}_{C\rtimes P}(d,y) \longrightarrow \text{よ}_{C}(d) \circ \pi_P$$

constructed in Lemma I.20 is pointwise injective and thus a monomorphism in $\mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}$. Hence, as g^* and $\mathbf{a}_K$ preserve finite limits, $\mathfrak{g}_d(y)$ is a monomorphism as desired. □

I.4.2 Relatively subcanonical sites

Finally, we generalise two results about subcanonical topologies to the canonical setting.

Recall that every natural transformation between morphisms of sites

$$\begin{tikzcd} (C,J) \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\beta"] & (\mathcal{D},K) \end{tikzcd}$$

induces a 2-cell of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{D},K) \arrow[r, bend left=50, "\mathbf{Sh}(G)"] \arrow[r, bend right=50, "\mathbf{Sh}(F)"'] \arrow[r, phantom, "\Downarrow\mathbf{Sh}(\beta)"] & \mathbf{Sh}(C,J). \end{tikzcd}$$

As observed in [63, Remark C2.3.5], the converse is also true if K is a *subcanonical topology*, i.e. the canonical functor $\ell_{\mathcal{D}}\colon \mathcal{D} \to \mathbf{Sh}(\mathcal{D},J)$ is fully faithful.

We will often encounter relative sites $\pi_Q\colon (\mathcal{D} \rtimes Q, K') \to (\mathcal{D}, J')$ where we would desire an analogous reasoning to apply, except that the topology K' is not truly subcanonical. For example, the topology $\tilde{J}_{\mathrm{can}}$ on the canonical fibration $\mathcal{E} \rtimes \mathcal{F}/f^*$ is not a subcanonical topology. See Remark II.15 for another related example.

We therefore desire an extension of the notion of subcanonical topology to the relative setting.

Definition I.26. Let $(C \rtimes P, K) \to (C, J)$ be a relative site. We will say that the topology K is *relatively subcanonical* if, for each object $d \in C$, the canonical natural transformation

$$\mathfrak{j}_d\colon P(d) \longrightarrow \mathbf{Sh}(C \rtimes P, K)/C^*_{\pi_P}\ell_C(d),$$

is full and faithful, where $\mathfrak{j}_d$ is induced as in Theorem I.21 by the commutative square

$$\begin{tikzcd} \mathbf{Sh}(C \rtimes P, K) \arrow[r, equal] \arrow[d, "C_{\pi_P}"'] & \mathbf{Sh}(C \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\ \mathbf{Sh}(C,J) \arrow[r, equal] & \mathbf{Sh}(C,J), \end{tikzcd}$$

i.e. j_d is the functor that sends an object $y \in P(d)$ to the arrow

$$\ell_{C \rtimes P}(d, y) \xrightarrow{\mathbf{a}_K(\text{ち}_{(d,y)})} C^*_{\pi_P} \ell_C(d).$$

Corollary I.27. *Let $(C \rtimes P, K)$ and $(\mathcal{D} \rtimes Q, K')$ be relative sites over, respectively, (C, J) and $(\mathcal{D}, J')$. Let*

$$\begin{tikzcd} (C \rtimes P, K) \arrow[r, "G"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & (\mathcal{D} \rtimes Q, K') \arrow[d, "\pi_Q"] \\ (C, J) \arrow[r, "H"] & (\mathcal{D}, J') \end{tikzcd} , \quad \begin{tikzcd} (C \rtimes P, K) \arrow[r, "G'"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & (\mathcal{D} \rtimes Q, K') \arrow[d, "\pi_Q"] \\ (C, J) \arrow[r, "H'"] & (\mathcal{D}, J') \end{tikzcd}$$

be two morphisms of relative sites and let $\gamma \colon H \Rightarrow H'$ be a natural transformation. If K' is a relatively subcanonical topology, then there is a bijection between the natural transformations $\beta \colon G \Rightarrow G'$ for which the 2-diagram

$$\begin{tikzcd} (C \rtimes P, K) \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow \beta"] \arrow[d, "\pi_P"'] & (\mathcal{D} \rtimes Q, K') \arrow[d, "\pi_Q"] \\ (C, J) \arrow[r, bend left, "H"] \arrow[r, bend right, "H'"'] \arrow[r, phantom, "\Downarrow \gamma"] & (\mathcal{D}, J') \end{tikzcd}$$

commutes, and the 2-cells of geometric morphisms $\beta' \colon \mathbf{Sh}(G) \Rightarrow \mathbf{Sh}(G')$ for which the 2-diagram

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{D} \rtimes Q, K') \arrow[r, bend left, "\mathbf{Sh}(G)"] \arrow[r, bend right, "\mathbf{Sh}(G')"'] \arrow[r, phantom, "\Downarrow \beta'"] \arrow[d, "C_{\pi_Q}"'] & \mathbf{Sh}(C \rtimes P, K) \arrow[d, "C_{\pi_P}"] \\ \mathbf{Sh}(\mathcal{D}, J') \arrow[r, bend left, "\mathbf{Sh}(H)"] \arrow[r, bend right, "\mathbf{Sh}(H')"'] \arrow[r, phantom, "\Downarrow \mathbf{Sh}(\gamma)"] & \mathbf{Sh}(C, J) \end{tikzcd}$$

commutes.

Proof. For notational convenience, let $\alpha \colon P \Rightarrow Q \circ H^{\mathrm{op}}$ and $\alpha' \colon P \Rightarrow Q \circ H'^{\mathrm{op}}$ be a pair of pseudo-natural transformations such that $G = H \rtimes \alpha$ and $G' = H' \rtimes \alpha'$.

One of the maps establishing the bijection comes from the functoriality of taking the induced geometric morphisms of a morphism of relative sites as demonstrated in Theorem I.21. It remains to show that every 2-cell of geometric morphisms

$\beta' \colon \mathbf{Sh}(G) \Rightarrow \mathbf{Sh}(G')$, for which $C_{\pi_P} * \beta' \simeq \mathbf{Sh}(\gamma) * C_{\pi_Q}$, induces a natural transformation $\beta \colon G \Rightarrow G'$, for which $\pi_P * \beta \simeq \gamma * \pi_Q$.

The pair of 2-cells $\beta' \colon \mathbf{Sh}(G) \Rightarrow \mathbf{Sh}(G')$ and $\mathbf{Sh}(\gamma) \colon \mathbf{Sh}(H) \Rightarrow \mathbf{Sh}(H')$ induce, by Theorem I.21, a natural transformation

$$\begin{tikzcd} \mathcal{C} \rtimes P \arrow[r, bend left=50, "G"{name=U}] \arrow[r, bend right=50, "G'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\beta''"] & \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/C^*_{\pi_Q} \end{tikzcd}$$

whose component at $(c,x) \in \mathcal{C} \rtimes P$ is the pair

$$\left(\begin{tikzcd} \mathbf{Sh}(H)^*\ell_{\mathcal{C}}(c) \arrow[d, "\mathbf{Sh}(\gamma)_{\ell_{\mathcal{C}}(c)}"] \\ \mathbf{Sh}(H')^*\ell_{\mathcal{C}}(c) \end{tikzcd}, \begin{tikzcd} \mathbf{Sh}(G)^*\ell_{\mathcal{C}\rtimes P}(c,x) \arrow[r, "\mathfrak{g}_c(x)"] \arrow[d, "\beta'_{\ell_{\mathcal{C}\rtimes P}(c,x)}"'] & C^*_{\pi_Q}\ell_{\mathcal{D}}H(c) \arrow[d, "\beta'_{C^*_{\pi_P}\ell_{\mathcal{C}}(c)}"] \\ \mathbf{Sh}(G')^*\ell_{\mathcal{C}\rtimes P}(x) \arrow[r, "\mathfrak{g}'_c(x)"] & C^*_{\pi_Q}\ell_{\mathcal{D}}H'(c) \end{tikzcd} \right).$$

We first note that the arrow $\mathbf{Sh}(\gamma)_{\ell_{\mathcal{C}}(c)}$ is induced by the arrow $H(c) \xrightarrow{\gamma_c} H'(c)$ in that there is a commuting square

$$\begin{tikzcd} \ell_{\mathcal{D}}H(c) \arrow[r, phantom, "="] \arrow[d, "\ell_{\mathcal{D}}(\gamma_c)"'] & \mathbf{Sh}(H)^*\ell_{\mathcal{C}}(c) \arrow[d, "\mathbf{Sh}(\gamma)_{\ell_{\mathcal{C}}(c)}"] \\ \ell_{\mathcal{D}}H'(c) \arrow[r, phantom, "="] & \mathbf{Sh}(H')^*\ell_{\mathcal{C}}(c). \end{tikzcd}$$

Next observe that there is an equality

$$\mathbf{Sh}(G)^*\ell_{\mathcal{C}\rtimes P}(c,x) = \ell_{\mathcal{D}\rtimes Q}G(c,x) = \ell_{\mathcal{D}\rtimes Q}H \rtimes \alpha(c,x) = \ell_{\mathcal{D}\rtimes Q}(H(c), \alpha_c(x)).$$

Similarly, we have that $\mathbf{Sh}(G')^*\ell_{\mathcal{C}\rtimes P}(c,x) = \ell_{\mathcal{D}\rtimes Q}(H'(c), \alpha'_c(x))$. By the definition of $\mathfrak{j}$, for each $(c,x) \in \mathcal{C} \rtimes P$ there is a commutative diagram

$$\begin{tikzcd} \ell_{\mathcal{D}\rtimes Q}(H(c),\alpha_c(x)) = \mathbf{Sh}(G)^*\ell_{\mathcal{C}\rtimes P}(c,x) \arrow[r, "\mathfrak{g}_c(x)"] \arrow[d, "\beta'_{\ell_{\mathcal{C}\rtimes P}(c,x)}"'] \arrow[r, bend left=30, "\mathfrak{j}_{H(c)}(\alpha_c(x))"] & C^*_{\pi_Q}\ell_{\mathcal{D}}H(c) \arrow[d, "\beta'_{C^*_{\pi_P}\ell_{\mathcal{C}}(c)} \simeq C^*_{\pi_Q}\ell_{\mathcal{D}}(\gamma_c)"] \\ \ell_{\mathcal{D}\rtimes Q}(H'(c),\alpha'_c(x)) = \mathbf{Sh}(G')^*\ell_{\mathcal{C}\rtimes P}(c,x) \arrow[r, "\mathfrak{g}'_c(x)"] \arrow[r, bend right=30, "\mathfrak{j}_{H'(c)}(\alpha'_c(x))"'] & C^*_{\pi_Q}\ell_{\mathcal{D}}H'(c). \end{tikzcd} \tag{I.vii}$$

Since the functor $\mathfrak{j}_{H(c)} \colon QH(c) \to \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/C^*_{\pi_Q}\ell_{\mathcal{D}}H(c)$ is fully faithful for all $c \in \mathcal{C}$, the pair $\left(\beta'_{\ell_{\mathcal{C}\rtimes P}(c,x)}, \beta'_{C^*_{\pi_P}\ell_{\mathcal{C}}(c)}\right)$ yields an arrow

$$(H(c), \alpha_c(x)) \xrightarrow{\beta_{(c,x)}} (H'(c), \alpha'_c(x)).$$

The naturality of β' ensures that this the component at (c, x) of a natural transformation $\beta\colon H \rtimes \alpha = G \Rightarrow G' = H' \rtimes \alpha'$. By the equivalence $\beta'_{C^*_{\pi_P}\ell_{\mathcal{C}}(c)} \simeq C^*_{\pi_Q}\ell_{\mathcal{D}}(\gamma_c)$ and the commutativity of the diagram (I.vii), the constructed natural transformation β makes the 2-diagram

$$\begin{tikzcd}
(\mathcal{C} \rtimes P, K) \arrow[rr, bend left, "G=H\rtimes\alpha"] \arrow[rr, bend right, "G'=H'\rtimes\alpha'"'] \arrow[rr, phantom, "\Downarrow\beta"] \arrow[d, "\pi_P"'] & & (\mathcal{D} \rtimes Q, K') \arrow[d, "\pi_Q"] \\
(\mathcal{C}, J) \arrow[rr, bend left, "H"] \arrow[rr, bend right, "H'"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & (\mathcal{D}, J')
\end{tikzcd}$$

commute up to isomorphism as required. □

Relative morphisms for cartesian valued relative sites. Given two cartesian categories $\mathcal{C}$ and $\mathcal{D}$ respectively endowed with Grothendieck topologies J and K, a left exact functor $F\colon \mathcal{C} \to \mathcal{D}$ defines morphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, K)$ if it sends J-covers to K-covers. In other words, by virtue of being left exact, F satisfies conditions (ii) to (iv) of Definition I.3 automatically. Recall from [107, Corollary 4.14] or Remark I.4 that the converse is true if K is a subcanonical topology. We will observe that this result generalises to the relative setting, thus yielding a more manageable description of the relative morphisms of sites in special cases.

We first recall the definition of a *modification*. Just as 1-categories $\mathcal{C}, \mathcal{D}$ yield 2-categorical functor categories $[\mathcal{C}, \mathcal{D}]$, given a pair of 2-categories $\mathfrak{A}, \mathfrak{B}$ the 2-functor category $[\mathfrak{A}, \mathfrak{B}]$ is naturally 3-categorical. These 3-cells

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\alpha \Downarrow \overset{\sigma}{\Rrightarrow} \Downarrow \beta"] & & \mathfrak{B}
\end{tikzcd}$$

are *modifications*. A modification consists of the data, for each object $A \in \mathfrak{A}$, a 2-cell

$$\begin{tikzcd}
FA \arrow[rr, bend left, "\alpha_A"] \arrow[rr, bend right, "\beta_A"'] \arrow[rr, phantom, "\Downarrow\sigma_A"] & & GA,
\end{tikzcd}$$

where the choice of 2-cell $\sigma_A\colon \alpha_A \Rightarrow \beta_A$ satisfies the coherence condition that, for every morphism $A \xrightarrow{f} B$ of $\mathfrak{A}$, the cylinder diagram

$$\begin{tikzcd}
FA \arrow[rr, "Ff"] \arrow[d, bend right=50, "\alpha_A"'] \arrow[d, bend left=50, "\beta_A"] \arrow[d, phantom, "\overset{\sigma_A}{\Longrightarrow}"] & \arrow[d, phantom, "\alpha_f \;\Downarrow \quad \Downarrow\; \beta_f"] & FB \arrow[d, bend right=50, "\alpha_B"'] \arrow[d, bend left=50, "\beta_B"] \arrow[d, phantom, "\overset{\sigma_B}{\Longrightarrow}"] \\
GA \arrow[rr, "Gf"'] & & GB
\end{tikzcd}$$

commutes.

Proposition I.28. *Let $(\mathcal{C} \rtimes P, K)$ and $(\mathcal{D} \rtimes Q, K')$ be relative sites over, respectively, $(\mathcal{C}, J)$ and $(\mathcal{D}, J')$, for which the associated pseudo-functors*

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathfrak{CAT} \text{ and } Q\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathfrak{CAT}$$

both factor through the category $\mathfrak{CART}$ of (large) cartesian categories and cartesian functors. Suppose further that K' is a relatively subcanonical topology. Then there is an equivalence of categories

$$\mathbf{RelMorph}\left(\begin{array}{ccc} (\mathcal{C} \rtimes P, K) & & (\mathcal{D} \rtimes Q, K') \\ \downarrow \pi_P & , & \downarrow \pi_Q \\ (\mathcal{C}, J) & & (\mathcal{D}, J') \end{array}\right) \simeq \mathbf{RelMorph}_{\mathrm{cart}}\big((\mathcal{C}, J, P, K), (\mathcal{D}, J', Q, K')\big),$$

where $\mathbf{RelMorph}_{\mathrm{cart}}\big((\mathcal{C}, J, P, K), (\mathcal{D}, J', Q, K')\big)$ is the category

(i) whose objects are pairs (F, a) consisting of a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a pseudo-natural transformation $a\colon P \Rightarrow Q \circ F^{\mathrm{op}}$, as in the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dr, "P"] \arrow[dd, "F^{\mathrm{op}}"'] & \\
\arrow[r, phantom, "\Downarrow a"] & \mathfrak{CART}, \\
\mathcal{D}^{\mathrm{op}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

such that

a) the functor F defines a morphism of sites $F\colon (\mathcal{C}, J) \to (\mathcal{D}, J')$,
b) for each object $c \in \mathcal{C}$, the component $a_c\colon P(c) \to Q(F(c))$ preserves finite limits,
c) and the induced functor $F \rtimes a\colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$ sends K-covers to K'-covers.

(ii) and whose arrows $(F, a) \xrightarrow{(\alpha,\sigma)} (F', a')$ are pairs consisting of a pseudo-natural transformation $\alpha\colon F \Rightarrow F'$ and a modification $\sigma\colon a \Rrightarrow a'$, as in the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dr, "P"] \arrow[dd, bend right=50, "F^{\mathrm{op}}"'] \arrow[dd, bend left=50, "F'^{\mathrm{op}}"] \arrow[dd, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\
\arrow[r, phantom, "a \Downarrow \overset{\sigma}{\Rrightarrow} \Downarrow a'"] & \mathfrak{CART}. \\
\mathcal{D}^{\mathrm{op}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

Proof. By [26, Corollary 2.2.6] (generalised to the setting with a change of base), the datum of a morphism of fibrations

$$\begin{tikzcd}
\mathcal{C} \rtimes P \arrow[r] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \rtimes Q \arrow[d, "\pi_Q"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

is equivalent in datum to a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a pseudo-natural transformation $a\colon P \Rightarrow Q \circ F^{\mathrm{op}}$, and furthermore a 2-cell between morphisms of fibrations

$$\begin{tikzcd}
\mathcal{C} \rtimes P \arrow[rr, bend left, "F \rtimes a"] \arrow[rr, bend right, "F' \rtimes a'"'] \arrow[rr, phantom, "\Downarrow"] \arrow[d, "\pi_P"'] & & \mathcal{D} \rtimes Q \arrow[d, "\pi_Q"] \\
P \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & Q
\end{tikzcd}$$

is equivalent in datum to a pair of a natural transformation $\alpha\colon F \Rightarrow F'$ and a modification $\sigma\colon a \Rrightarrow a'$.

Thus, to exhibit the desired equivalence it suffices to demonstrate an equivalence on objects, i.e. we wish to show that the pair

$$(F, F \rtimes a)\colon \left[(\mathcal{C} \rtimes P, K) \xrightarrow{\pi_P} (\mathcal{C}, J)\right] \longrightarrow \left[(\mathcal{D} \rtimes Q, K') \xrightarrow{\pi_Q} (\mathcal{D}, J')\right]$$

defines a morphism of relative sites if and only if

$$(F, a) \in \mathbf{RelMorph}_{\mathrm{cart}}\Big((\mathcal{C}, J, P, K), (\mathcal{D}, J', Q, K')\Big).$$

We deduce that it is enough to show that $F \rtimes a\colon (\mathcal{C} \rtimes P, K) \to (\mathcal{D} \rtimes Q, K')$ satisfies conditions (ii) to (iv) from Definition I.3 if and only if, for each $c \in \mathcal{C}$, the component $a_c\colon P(c) \to Q(F(c))$ preserves finite limits.

We begin with the 'left to right' proof. Suppose that $(F, F \rtimes a)$ is a morphism of relative sites. By Lemma I.19, there is a morphism of relative topoi

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[d, "C_{\pi_P}"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{D} \rtimes Q, K') \arrow[l, "\mathbf{Sh}(F \rtimes a)"'] \arrow[d, "C_{\pi_Q}"] \\
\mathbf{Sh}(\mathcal{C}, J) & \mathbf{Sh}(\mathcal{D}, J'). \arrow[l, "\mathbf{Sh}(F)"']
\end{tikzcd}$$

Let 1_c denote the terminal object of $P(c)$ for an object $c \in \mathcal{C}$. For any other object $(e, x) \in \mathcal{C} \rtimes P$, there is an arrow $(c', x) \xrightarrow{(f,g)} (c, 1_c) \in \mathcal{C} \rtimes P$, which must necessarily factor as

$$(e, x) \xrightarrow{(\mathrm{id}_e, g)} (e, 1_e) \cong (e, P(f)(1_c)) \xrightarrow{(f, \mathrm{id}_{P(f)(1_c)})} (c, 1_c),$$

if and only if there is an arrow $e \xrightarrow{f} c \in \mathcal{C}$, and so we deduce that there is an isomorphism $\text{よ}_{\mathcal{C}\rtimes P}(c, 1_c) \cong \text{よ}_{\mathcal{C}}(c)$. Thus, we deduce further that

$$\begin{aligned} \mathcal{C}^*_{\pi_P}\ell_{\mathcal{C}}(c) &= \mathbf{a}_K(\mathbf{a}_J\,\text{よ}_{\mathcal{C}}(c) \circ \pi_P), \\ &\cong \mathbf{a}_K(\,\text{よ}_{\mathcal{C}}(c) \circ \pi_P), \\ &\cong \mathbf{a}_K(\,\text{よ}_{\mathcal{C}\rtimes P}(c, 1_c)) = \ell_{\mathcal{C}\rtimes P}(c, 1_c), \end{aligned}$$

and similarly that $\mathcal{C}^*_{\pi_Q}\ell_{\mathcal{D}}(d) \cong \ell_{\mathcal{D}\rtimes Q}(d, 1_d)$. Therefore, by chasing the object $c \in \mathcal{C}$ around the commuting diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\ell_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\ell_{\mathcal{D}}"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "\mathbf{Sh}(F)^*"] \arrow[d, "\mathcal{C}^*_{\pi_P}"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{D}, J') \arrow[d, "\mathcal{C}^*_{\pi_Q}"] \\ \mathbf{Sh}(\mathcal{C} \rtimes P, K) \arrow[r, "\mathbf{Sh}(F\rtimes a)^*"] & \mathbf{Sh}(\mathcal{D} \rtimes Q, K'), \end{tikzcd}$$

we conclude that $\mathbf{Sh}(F \rtimes a)^*\ell_{\mathcal{C}\rtimes P}(c, 1_c) \cong \ell_{\mathcal{D}\rtimes Q}(F(c), 1_{F(c)})$.

The inclusion $P(c) \hookrightarrow \mathcal{C} \rtimes P/(c, 1_c)$,

$$x \mapsto (c, x) \xrightarrow{(\mathrm{id}_c, !)} (c, 1_c),$$

can easily be shown to preserve finite limits. Therefore, in the commutative diagram

$$\begin{tikzcd} P(c) \arrow[r, hook] \arrow[dd, "a_c"'] & \mathcal{C} \rtimes P/(c, 1_c) \arrow[r, "\ell_{\mathcal{C}\rtimes P}/(c,1_c)"] \arrow[dd, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, K)/\ell_{\mathcal{C}\rtimes P}(c, 1_c) \arrow[d, "\mathbf{Sh}(F\rtimes a)^*/\ell_{\mathcal{C}\rtimes P}(c,1_c)"] \\ & & \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/\mathbf{Sh}(F \rtimes a)^*\ell_{\mathcal{C}\rtimes P}(c, 1_c) \arrow[d, phantom, "\wr"] \\ Q(F(c)) \arrow[r, hook] \arrow[drr, bend right, "\mathfrak{j}_{F(c)}"'] & \mathcal{D} \rtimes Q/(F(c), 1_{F(c)}) \arrow[r, "\ell_{\mathcal{D}\rtimes Q}/(F(c),1_{F(c)})"'] & \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/\ell_{\mathcal{D}\rtimes Q}(F(c), 1_{F(c)}) \arrow[d, phantom, "\wr"] \\ & & \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/\mathcal{C}^*_{\pi_Q}\ell_{\mathcal{D}}(F(c)), \end{tikzcd}$$

the composite $\mathfrak{j}_{F(c)} \circ a_c \colon P(c) \to \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/\mathcal{C}^*_{\pi_Q}\ell_{\mathcal{D}}(F(c))$ preserves finite limits too. By hypothesis, K' is a relatively subcanonical topology, meaning that the composite

$$\mathfrak{j}_{F(c)} \colon Q(F(c)) \longrightarrow \mathbf{Sh}(\mathcal{D} \rtimes Q, K')/\mathcal{C}^*_{\pi_Q}\ell_{\mathcal{D}}(F(c))$$

is fully faithful. In particular, $\mathfrak{j}_{F(c)}$ reflects finite limits. Hence, if $(F, F \rtimes a)$ is a morphism of relative sites, then a_c must preserve finite limits as required.

For the converse direction, it is easily shown directly that if each component $a_c \colon P(c) \to Q(F(c))$ preserves finite limits, then $F \rtimes a \colon (\mathcal{C} \rtimes P, K) \to (\mathcal{D} \rtimes Q, K')$ satisfies conditions (ii) to (iv) from Definition I.3. We complete the proof that condition (ii) of Definition I.3 is satisfied. The others follow a similar pattern.

Let (d, x) be an object of $\mathcal{D} \rtimes Q$. Since $F \colon (\mathcal{C}, J) \to (\mathcal{D}, J')$ is also a morphism of sites, there is a J'-covering family of arrows

$$S = \left\{ d_i \xrightarrow{f_i} d \,\middle|\, i \in I \right\}$$

for which each d_i has an arrow $d_i \xrightarrow{g_i} F(c_i)$. As $\pi_Q \colon (\mathcal{D} \rtimes Q, K') \to (\mathcal{D}, J')$ has the cover lifting property, the family S is lifted to a K'-covering family of arrows

$$\left\{ (d_i, x_i) \xrightarrow{(f_i,h_i)} (d,x) \,\middle|\, i \in I \right\}.$$

We need only conclude that, for each $i \in I$, there is a morphism

$$(d_i, x_i) \xrightarrow{(g_i,!)} (F(c_i), 1_{F(c_i)}) \cong (F(c_i), a_{c_i}(1_{c_i}))$$

to realise that Definition I.3(ii) is satisfied. □

Chapter II

Internal locale theory

Pointless topology. By and large, the topologically interesting data of a space or a continuous map is contained in the algebra of open sets and the inverse image map. This prompted the shift to 'pointfree' topology, as exposited in [61], [62], where *locales* replace spaces and *locale morphisms* replace continuous maps.

Internal locales. Abstracting further, each topos $\mathcal{E}$ has a rich internal language in which locale theory can be internalised. The *internal locales* of a topos can be studied by re-externalising the internal constructions, treating them as relative sites as studied in Chapter I. As suggested by Examples 0.5, the internal locales of many topoi can be of interest outside of topos theory.

Therefore, for applications it is beneficial to have a well-developed dictionary externalising notions for internal locales. Examples of external accounts of internal locale theory can be found in [68], [63, §C1.6] and [24].

Contributions of this chapter. Akin to [24], we study internal locales in the language of relative sites, as reviewed in Chapter I. We aim to recreate an internal version of the treatment of localic topoi and their morphisms found in [79, §IX]. We will observe that the most commonly considered properties of internal locale morphisms admit satisfying externalisations, namely that:

(i) surjections of internal locales,

(ii) embeddings of internal sublocales,

(iii) and the co-frame operations on the co-frame of internal sublocales

can all be computed *'pointwise'*.

Overview. The chapter is divided as follows.

(A) A brief recount of the basic theory of (set-based) locales is given in Section II.1. Further results from locale theory are introduced when needed.

(B) In Section II.2, a review is given of the classification of internal locales for the topos $\mathbf{Sh}(\mathcal{C}, J)$ as established in [68, Proposition VI.2.2] and [24, Proposition 5.10]. We also recall the construction of the relative topos of *internal sheaves* $\mathbf{Sh}(\mathbb{L}) \to \mathcal{E}$ on an internal locale $\mathbb{L}$ of $\mathcal{E}$ as described in [63, Examples C2.5.8(c)] and [24, Definition 5.2].

(C) Some examples of internal locales whose base categories are not cartesian are presented in Section II.3, including internal locales of topoi of monoid actions.

(D) Our study of internal locale morphisms begins in Section II.4. It is well-known (see [68, §VI.5], [59, §2], or [24, Corollary 3.5]) that, given internal locales $\mathbb{L}$ and $\mathbb{L}'$ of a topos $\mathcal{E} \simeq \mathbf{Sh}(C, J)$, there is an equivalence between internal locale morphisms $\mathfrak{f} \colon \mathbb{L} \to \mathbb{L}'$ and geometric morphisms g for which the diagram

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{L}) \arrow[rr, "g"] \arrow[dr] & & \mathbf{Sh}(\mathbb{L}') \arrow[dl] \\ & \mathcal{E} & \end{tikzcd}$$

commutes. We give a direct using Corollary I.23.

It is further shown in Proposition II.28, that the geometric morphism $\mathbf{Sh}(\mathfrak{f})$ induced by an internal locale morphism $\mathfrak{f} \colon \mathbb{L} \to \mathbb{L}'$ is *surjective* if and only if each component $\mathfrak{f}_c \colon \mathbb{L}(c) \to \mathbb{L}'(c)$, for $c \in C$, is a surjective locale morphism, i.e. surjections of internal locales are computed 'pointwise'.

(E) The internal locale morphisms that induce embeddings of subtopoi are the subject of Section II.5. We show that *internal locale embeddings* coincide with 'pointwise' locale embeddings. We also introduce the notion of an *internal nucleus* on an internal locale, a mild generalisation of a *Lawvere-Tierney topology*, and show that these too correspond bijectively with internal sublocale embeddings.

(F) Finally in Section II.6, we study the co-frame $\mathrm{Sub}_{\mathbf{Topos}}(\mathbf{Sh}(\mathbb{L}))$ of subtopoi of $\mathbf{Sh}(\mathbb{L})$ (see [63, §A4.5] or [22, §4]). We show that the co-frame operations of $\mathrm{Sub}_{\mathbf{Topos}}(\mathbf{Sh}(\mathbb{L}))$ can be computed 'pointwise' via the co-frame operations on $\mathrm{Sub}_{\mathbf{Loc}}(\mathbb{L}(c))$, the co-frame of sublocales of $\mathbb{L}(c)$, for each $c \in C$.

II.1 Background on locales

If we forget about points, topology is the study of algebras of open sets $\mathcal{O}(X)$ and the action of continuous maps $f^{-1} \colon \mathcal{O}(X) \to \mathcal{O}(Y)$ on these open sets. The notions of frame and frame homomorphism capture these purely algebraic aspects of topology.

Definition II.1. A *frame* L is a complete lattice satisfying, for each subset $\{ U_i \mid i \in I \} \subseteq L$ and $V \in L$, the infinite distributivity law

$$V \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} V \wedge U_i.$$

A *frame homomorphism* is any map between frames that preserves arbitrary joins and finite meets. We denote the resultant category by **Frm**.

Our motivating examples, the algebra of opens $\mathcal{O}(X)$ of a topological space X and the inverse image map $f^{-1} \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ of a continuous map $f \colon X \to Y$, are both examples of, respectively, a frame and a frame homomorphism. To strengthen the analogy with topological spaces, one often works with the category of *locales* $\mathbf{Loc} \simeq \mathbf{Frm}^{\mathrm{op}}$ instead.

Notation II.2. For a locale morphism $f\colon L \to K$, we will use $f^{-1}\colon K \to L$ to denote the corresponding frame homomorphism. Additionally, each frame homomorphism $f^{-1}\colon K \to L$ has a right adjoint $f_*\colon L \to K$, since K is complete.

Frames are equivalently *complete Heyting algebras* (see [97, Proposition 7.3.2, Appendix 1]). The Heyting implication in a frame L is given by

$$U \to V = \bigvee \{ W \in L \mid W \wedge U \leqslant V \}.$$

However, frame homomorphisms need not preserve the Heyting implication.

Definition II.3 (§V.1 [68])**.** A frame homomorphism $f\colon L \to K$ is said to be *open* if either of the following equivalent conditions are satisfied:

(i) $f\colon L \to K$ is a complete Heyting algebra homomorphism,

(ii) $f^{-1}\colon K \to L$ has a left adjoint $\exists_f$ which satisfies the *Frobenius condition*:

$$\exists_f(U \wedge f^{-1}(V)) = \exists_f(U) \wedge V,$$

for all $U \in L$ and $V \in K$.

Open frame homomorphisms generalise open continuous maps (as can be seen by [79, Proposition IX.7.5]). We will use $\mathbf{Frm}_{\text{open}}$ to denote the category of frames and open frame homomorphisms, and $\mathbf{Loc}_{\text{open}}$ to denote the opposite category $\mathbf{Frm}_{\text{open}}^{\text{op}}$.

In [54], Isbell promulgates the use of **Loc** as a constructive alternative to topological spaces, since many desirable properties hold (constructively) for locales whose topological analogies do not (see, for instance, [56]).

II.2 Internal locales

An *internal locale* of a topos $\mathcal{E}$ is an object that, according to the internal language of $\mathcal{E}$, carries the structure of a locale (equivalently, a complete Heyting algebra).

Examples II.4. (i) Unsurprisingly, the internal locales of **Sets**, the topos of sets, are just locales.

(ii) (Theorem C1.6.3 [63]) An internal locale of a localic topos $\mathbf{Sh}(X)$ is a locale morphism $Y \to X$.

(iii) For any topos $\mathcal{E}$, the subobject classifier $\Omega_{\mathcal{E}}$ is an internal locale of $\mathcal{E}$. In fact, we will see in Corollary II.26 that $\Omega_{\mathcal{E}}$ is the terminal internal locale in $\mathcal{E}$.

More examples will be presented in Section II.3.

We devote this section to a review of the external treatment of internal locales: that is, given a Grothendieck topos $\mathcal{E}$ with a site of definition (C, J), a classification for which J-sheaves $\mathbb{L}\colon C^{\text{op}} \to \mathbf{Sets}$ correspond to internal locales of $\mathcal{E} \simeq \mathbf{Sh}(C, J)$. An externalised treatment of internal locales can be found in [68, §VI] and [63, §C1.6] for the special case when C is *cartesian* (i.e. C has all finite limits). When C is non-cartesian, [24, §5] establishes a classification of internal locales of $\mathbf{Sh}(C, J)$, which will form the basis of our treatment.

Notation II.5. Given a functor $\mathbb{L}\colon C^{\text{op}} \to \mathbf{Frm}_{\text{open}}$, an object c and an arrow g of C, when there is no confusion we will use the shorthand $\mathbb{L}_c$ for $\mathbb{L}(c)$, g^{-1} for $\mathbb{L}(g)$ and $\exists_g$ for the left adjoint to $\mathbb{L}(g)$.

II.2.1 Internal locales of a presheaf topos

We begin with an overview of the classification of internal locales of a presheaf topos $\mathbf{Sets}^{C^{op}}$, where C is an arbitrary category, as calculated in [24, §5]. We will also observe that this characterisation subsumes the previous characterisation of Joyal and Tierney [68, §VI] for internal locales over a cartesian base category.

Localic geometric morphisms. The 'keystone' property used in [24] for the classification of internal locales is the connection between internal locales and localic geometric morphisms.

Definition II.6. A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is *localic* if every object F of $\mathcal{F}$ is a subquotient of $f^*(E)$ for some $E \in \mathcal{E}$, i.e. there exists $F' \in \mathcal{F}$ and a diagram

$$F \twoheadleftarrow F' \rightarrowtail f^*(E).$$

Localic geometric morphisms $f : \mathcal{F} \to \mathcal{E}$ correspond bijectively (up to isomorphism) to internal locales of $\mathcal{E}$ via the following result.

Theorem II.7 (Theorem 5.37 [57] or Lemma 1.2 [59], cf. also Proposition 4.2 [24])**.** *For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the following are equivalent:*

(i) f is a localic geometric morphism,

(ii) $\mathcal{F}$ is the topos of internal sheaves on an internal locale of $\mathcal{E}$, and moreover this internal locale can be taken as $f_*(\Omega_{\mathcal{F}})$.

This bijection can be visualised with the 'bridge' diagram

$$\begin{tikzcd}
& \mathcal{F} \arrow[d, "f"] & \\
& \begin{array}{c} \mathcal{E} \simeq \mathbf{Sh}(C, J) \\ \textit{localic morphism} \end{array} \arrow[dl, dashed, bend right, <->] \arrow[dr, dashed, bend left, <->] & \\
\begin{array}{c} f_*(\Omega_{\mathcal{F}}) \\ \textit{direct image of} \\ \textit{subobject classifier} \end{array} & & \begin{array}{c} \mathbb{L} \in \mathcal{E} \\ \textit{internal locale.} \end{array}
\end{tikzcd}$$

Let $\mathbb{L}$ be an internal locale of $\mathcal{E} \simeq \mathbf{Sh}(C, J)$. It appears as the direct image of the subobject classifier $f_*(\Omega_{\mathcal{F}}) \cong \mathbb{L}$ for some localic geometric morphism $f : \mathcal{F} \to \mathcal{E}$. Considered as a sheaf $f_*(\Omega_{\mathcal{E}}) : \mathcal{E}^{op} \to \mathbf{Sets}$ on the canonical site $(\mathcal{E}, J_{can})$ for $\mathcal{E}$, there is the chain of isomorphisms

$$\begin{aligned} f_*(\Omega_{\mathcal{F}}) &\cong \mathcal{E}(-, f_*(\Omega_{\mathcal{F}})), \\ &\cong \mathcal{F}(f^*-, \Omega_{\mathcal{F}}), \\ &\cong \mathrm{Sub}_{\mathcal{F}}(f^*-) \end{aligned}$$

(here, the first isomorphism is by the Yoneda lemma). Hence, by composing with the canonical morphism $\ell_C : C \to \mathbf{Sh}(C, J)$, we obtain the isomorphism of J-sheaves:

$$\mathbb{L} \cong \mathrm{Sub}_{\mathcal{F}}(f^* \circ \ell_C -) : C^{\mathrm{op}} \to \mathbf{Sets}. \tag{II.i}$$

Thus, we can observe some basic facts about the internal locale $\mathbb{L}$:

(i) for each object c of C, $\mathbb{L}(c)$ is a complete Heyting algebra, or frame, by [79, Proposition III.8.1];
(ii) for each arrow $f : c \to d$ of C, $\mathbb{L}(f) : \mathbb{L}(d) \to \mathbb{L}(c)$ is an open frame homomorphism by [79, Proposition III.8.2].

Although not every such functor $\mathbb{L}' : C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ will yield an internal locale, it is possible to characterise when they do.

The relative Beck-Chevalley condition. Given any functor

$$\mathbb{L} : C^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}},$$

we define $K_{\mathbb{L}}$ as the function that assigns to each object (d, V) of $C \rtimes \mathbb{L}$ the collection $K_{\mathbb{L}}(c)$ of sieves in $C \rtimes \mathbb{L}$ that contain small families

$$\left\{ (c_i, U_i) \xrightarrow{f_i} (d, V) \,\middle|\, i \in I \right\}$$

such that $V = \bigvee_{i \in I} \exists_{f_i} U_i$.

Thus defined, $K_{\mathbb{L}}$ is not necessarily a Grothendieck topology on $C \rtimes \mathbb{L}$. The assignment of sieves $K_{\mathbb{L}}$ clearly satisfies the maximality and transitivity conditions, but $K_{\mathbb{L}}$ does not always satisfy the stability condition (see [79, Definition III.2.1]).

When $K_{\mathbb{L}}$ does define a Grothendieck topology, the topos $\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})$ is also definable and moreover the geometric morphism

$$C_{\pi_{\mathbb{L}}} : \mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}}) \longrightarrow \mathbf{Sets}^{C^{\mathrm{op}}},$$

induced by the projection $\pi_{\mathbb{L}} : C \rtimes \mathbb{L} \to C$, considered as a comorphism of sites

$$\pi_{\mathbb{L}} : (C \rtimes \mathbb{L}, K_{\mathbb{L}}) \longrightarrow (C, J_{\mathrm{triv}}),$$

is localic by [23, Proposition 7.11]. Since each fibre has a top element, the functor $\pi_{\mathbb{L}}$ has a left adjoint $t_{\mathbb{L}} : C \to C \rtimes \mathbb{L}$ that sends $c \in C$ to the object $(c, \top_c) \in C \rtimes \mathbb{L}$. Therefore, the direct image functor $C_{\pi_{\mathbb{L}}*}$ of the induced geometric morphism acts as $- \circ t_{\mathbb{L}}$ by [79, Theorem VII.10.4]. It is not now difficult to calculate, as is done in [24, §5], that

$$\mathbb{L} \cong C_{\pi_{\mathbb{L}}*}(\Omega_{\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})}) \cong \Omega_{\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})} \circ t_{\mathbb{L}}.$$

Remark II.8. In the language of [24, Definition 5.1], if $K_{\mathbb{L}}$ does define a Grothendieck topology on $C \rtimes \mathbb{L}$, then the site $(C \rtimes \mathbb{L}, K_{\mathbb{L}})$ is an example of an *existential site*, $K_{\mathbb{L}}$ is an *existential topology* and $\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})$ is an *existential topos*. Existential topoi will be discussed in more detail in Section III.3.1.

Thus, we arrive at the classification of internal locales in the topos $\mathbf{Sets}^{C^{\mathrm{op}}}$ established in [24, §5].

Definition II.9 (Definition 5.1(e)(i) [24])**.** A functor $\mathbb{L}\colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ is said to satisfy the *relative Beck-Chevalley condition* if, given an arrow $e \xrightarrow{h} d$ of C, and a sieve S of $C \rtimes \mathbb{L}$ on the object (d, V) for which $V = \bigvee_{f \in S} \exists_f U$, then

$$h^{-1}(V) = \bigvee_{g \in h^*(S)} \exists_g W,$$

where $h^*(S)$ denotes the sieve on $(e, h^{-1}(V))$ containing those arrows $(c, W) \xrightarrow{g} (e, h^{-1}(V))$ for which the composite

$$(c, W) \xrightarrow{\ g\ } (e, h^{-1}(V)) \xrightarrow{\ h\ } (d, V)$$

is in S.

Theorem II.10 (Proposition 5.10 [24])**.** *Let* $\mathbb{L}\colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ *be a functor. The following are equivalent:*

(i) $\mathbb{L}$ *is an internal locale of* $\mathbf{Sets}^{C^{\mathrm{op}}}$*,*

(ii) $\mathbb{L}$ *satisfies the relative Beck-Chevalley condition,*

(iii) $K_{\mathbb{L}}$ *is a Grothendieck topology on* $C \rtimes \mathbb{L}$*.*

Definition II.11 (Theorem 5.1 [24])**.** Let $\mathbb{L}$ be an internal locale of $\mathbf{Sets}^{C^{\mathrm{op}}}$. The topos $\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})$ is called the *topos of internal sheaves* (or just *topos of sheaves*) on $\mathbb{L}$.

Remark II.12. Let $\mathbb{L}$ be an internal locale of $\mathbf{Sets}^{C^{\mathrm{op}}}$. It is not hard to recognise that the isomorphism of frames

$$\{ V \in \mathbb{L}_c \mid V \leqslant \top_c \} \cong \mathbb{L}_c \cong C_{\pi_{\mathbb{L}*}}\left(\Omega_{\mathbf{Sh}(\mathbb{L})}\right)(c) \cong \Omega_{\mathbf{Sh}(L)} \circ t_{\mathbb{L}}(c) \cong \Omega_{\mathbf{Sh}(\mathbb{L})}(c, \top_c).$$

can be extended so that, for each object (c, U) of $C \rtimes \mathbb{L}$, there is an isomorphism

$$\{ V \in \mathbb{L}_c \mid V \leqslant U \} \cong \Omega_{\mathbf{Sh}(\mathbb{L})}(c, U),$$

and that, for each morphism $(c, U) \xrightarrow{f} (d, W)$ of $C \rtimes \mathbb{L}$, the transition map

$$\Omega_{\mathbf{Sh}(\mathbb{L})}(f)\colon \Omega_{\mathbf{Sh}(\mathbb{L})}(d, W) \to \Omega_{\mathbf{Sh}(\mathbb{L})}(c, U)$$

sends $V \in \Omega_{\mathbf{Sh}(\mathbb{L})}(d, W)$ to $f^{-1}(V) \wedge U \in \Omega_{\mathbf{Sh}(\mathbb{L})}(c, U)$.

The classification of internal locales of $\mathbf{Sets}^{C^{\mathrm{op}}}$ originally given in [68, Proposition VI.2.2] for the case where C is a cartesian category can be recovered via the above classification by noting, as is done in [24, Proposition 5.3], that the Beck-Chevalley and relative Beck-Chevalley conditions coincide when C has all finite limits (in fact, a study of the proof of [24, Proposition 5.3] reveals that only pullbacks are necessary).

Corollary II.13 (Proposition 5.3 & Corollary 5.4 [24]). *Let C be a category with all pullbacks. A functor $\mathbb{L}\colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ satisfies the relative Beck-Chevalley condition, and thus defines an internal locale of $\mathbf{Sets}^{C^{\mathrm{op}}}$, if and only if $\mathbb{L}$ satisfies the Beck-Chevalley condition: for each pullback square*

$$\begin{tikzcd} c \times_e d \arrow[r, "g"] \arrow[d, "k"'] & d \arrow[d, "h"] \\ c \arrow[r, "f"] & e \end{tikzcd}$$

of C, the square

$$\begin{tikzcd} \mathbb{L}_{c\times_e d} \arrow[r, "\exists_g"] & \mathbb{L}_d \\ \mathbb{L}_c \arrow[u, "k^{-1}"] \arrow[r, "\exists_f"] & \mathbb{L}_e \arrow[u, "h^{-1}"'] \end{tikzcd}$$

commutes.

The topology $K_{\mathbb{L}}$. We complete this discussion with some observations concerning the Grothendieck topology $K_{\mathbb{L}}$.

Proposition II.14 (Remark 5.1 [24]). *Let $\mathbb{L}$ be an internal locale of $\mathbf{Sets}^{C^{\mathrm{op}}}$. The Grothendieck topology $K_{\mathbb{L}}$ on $C \rtimes \mathbb{L}$ is generated by the following two species of covering families:*

(A) $\left\{ (c, U) \xrightarrow{f} (d, \exists_f U) \right\}$ *for each arrow $c \xrightarrow{f} d$ of C and $U \in \mathbb{L}_c$,*

(B) $\left\{ (c, U_i) \xrightarrow{\mathrm{id}_c} (c, \bigvee_{i\in I} U_i) \,\middle|\, i \in I \right\}$ *for each object c of C and each $\{ U_i \mid i \in I \} \subseteq \mathbb{L}_c$.*

Proof. We immediately have that both species are $K_{\mathbb{L}}$-covering. For the converse, note that, given a $K_{\mathbb{L}}$-covering sieve S on (d, V), each morphism $(c, U) \xrightarrow{f} (d, V)$ of S can be written as the composite

$$(c, U) \xrightarrow{\ f\ } (d, \exists_f U) \xrightarrow{\ \mathrm{id}_d\ } \left(d, \bigvee_{f\in S} \exists_f U \right) = (d, V).$$

Hence, any Grothendieck topology J for which both species (A) and (B) are J-covering contains the Grothendieck topology $K_{\mathbb{L}}$. □

Remark II.15. Let $\mathbb{L}$ be an internal locale of $\mathbf{Sets}^{C^{\mathrm{op}}}$.

(i) We have refrained from naming the Grothendieck topology $K_{\mathbb{L}}$ the 'canonical topology' to avoid confusion, despite it being a generalisation of the canonical topology on a locale. Unlike a locale L of **Sets**, the Grothendieck topology $K_{\mathbb{L}}$ is *not* necessarily a subcanonical topology. Recall from [63, p. 542-3, §C1.2] that a Grothendieck topology J on a category $\mathcal{D}$ is subcanonical only if every J-covering sieve S on an object D is *effective-epimorphic*, in the sense that D is the colimit of the (potentially large) diagram

$$S \hookrightarrow \mathcal{D}/D \xrightarrow{\ U\ } \mathcal{D},$$

where $U\colon \mathcal{D}/D \to \mathcal{D}$ is the forgetful functor. Observe, however, that the sieve generated by a $K_{\mathbb{L}}$-covering family

$$\left\{ (c, U) \xrightarrow{f} (d, \exists_f U) \right\}$$

of species (A) is not effective-epimorphic for any non-invertible arrow f of C since the colimit in $C \rtimes \mathbb{L}$ is given by (c, U).

(ii) In contrast, the topology $K_{\mathbb{L}}$ is a *relatively subcanonical* topology in the sense of Definition I.26 since, for each $c \in C$, there is an isomorphism

$$\mathbb{L}(c) \cong \mathrm{Sub}_{\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})}(C^*_{\pi_{\mathbb{L}}} \text{よ}_C(c)) \subseteq \mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})/C^*_{\pi_{\mathbb{L}}} \text{よ}_C(c),$$

i.e. the functor $\mathrm{j}_c\colon \mathbb{L}(c) \to \mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}})/C^*_{\pi_{\mathbb{L}}} \text{よ}_C(c)$ is full and faithful.

II.2.2 Internal locales of sheaf topoi

In this final subsection, it is demonstrated how a classification of the internal locales of the presheaf topos $\mathbf{Sets}^{C^{op}}$ yields a classification of the internal locales of the sheaf topos $\mathbf{Sh}(C, J)$.

Let (C, J) be a Grothendieck site. The embedding $\mathbf{Sh}(C, J) \rightarrowtail \mathbf{Sets}^{C^{op}}$ is a localic geometric morphism (see [63, Example A4.6.2(a)]), and thus, for any localic geometric morphism $\mathcal{F} \to \mathbf{Sh}(C, J)$, the composite

$$\mathcal{F} \to \mathbf{Sh}(C, J) \rightarrowtail \mathbf{Sets}^{C^{op}}$$

is still localic since localic geometric morphisms are closed under composition (see [59, Lemma 1.1]). Therefore, our understanding of the internal locales of the presheaf topos $\mathbf{Sets}^{C^{op}}$ can be leveraged to describe the internal locales of $\mathbf{Sh}(C, J)$.

Lemma II.16 (Proposition 5.10 [24], Corollary C1.6.10[63])**.** *Let $\mathbb{L}\colon C^{op} \to \mathbf{Frm}_{open}$ be a functor indexed over a category C with a Grothendieck topology J. The following are equivalent:*

(i) $\mathbb{L}$ is an internal locale of $\mathbf{Sh}(C, J)$,

(ii) $\mathbb{L}$ is an internal locale of $\mathbf{Sets}^{C^{op}}$ and a J-sheaf,

(iii) $K_{\mathbb{L}}$ is stable and contains the Giraud topology $J_{\pi_{\mathbb{L}}}$,

(iv) $K_{\mathbb{L}}$ is stable and there exists a factorisation

$$\begin{tikzcd} & \mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}}) \arrow[dl, dashed] \arrow[d, "C_{\pi_{\mathbb{L}}}"] \\ \mathbf{Sh}(C, J) \arrow[r, tail] & \mathbf{Sets}^{C^{op}}. \end{tikzcd}$$

Proof. The equivalence of statements (i) and (ii) is a consequence of the fact that the direct image of a geometric morphism (in this case the inclusion $\mathbf{Sh}(C, J) \hookrightarrow \mathbf{Sets}^{C^{op}}$) preserves internal locales (see p. 528 [63, §C1.6], c.f. [63, Corollary C1.6.10] as well). The equivalence of (ii) and (iii) is proved in [24, Proposition 5.10] (cf. Remark 5.3(b) [24] too). The final equivalence of (iii) and (iv) follows by definition of the Giraud topology. ∎

II.3 Examples of internal locales

We now consider some examples of internal locales over non-cartesian base categories.

II.3.1 Gluing internal locales

What can prevent a functor $\mathbb{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ from being an internal locale of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$? What goes wrong when $K_{\mathbb{L}}$ is not stable? We give an example of such a functor, over a category $\mathcal{C}$ without all pullbacks, which is not an internal locale, despite $\mathbb{L}$ satisfying the Beck-Chevalley condition for those pullbacks in $\mathcal{C}$ that do exist. Inspired by this counterexample, we develop in Corollary II.18 a method for identifying the internal locales of the presheaf topos $\mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}}$ when $\mathcal{D}$ is obtained by 'gluing' certain constituent subcategories together.

Example II.17. Let L be any locale in **Sets**. For any category $\mathcal{C}$ with pullbacks, the constant functor $\mathbb{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ for L, i.e. $\mathbb{L}(c) = L$ and $\mathbb{L}(f) = \mathrm{id}_L$ for all objects c and arrows f of $\mathcal{C}$, satisfies the Beck-Chevalley condition and so defines an internal locale of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

Now consider the category

$$\begin{tikzcd} \bullet_1 \arrow[r, "f"] \arrow[loop below, "\mathrm{id}_1"] & \bullet_2 \arrow[loop below, "\mathrm{id}_2"] & \bullet_3 \arrow[l, "g"'] \arrow[loop below, "\mathrm{id}_3"] \end{tikzcd}$$

with all arrows displayed (we will refer to it as $\bullet \to \bullet \leftarrow \bullet$), which clearly lacks a pullback for the diagram

$$\begin{tikzcd} & \bullet_3 \arrow[d, "g"] \\ \bullet_1 \arrow[r, "f"] & \bullet_2. \end{tikzcd}$$

The constant functor $\mathbb{L}\colon (\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ for a non-trivial locale L is not an internal locale of $\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}$. We can observe that the relative Beck-Chevalley condition fails. For instance, the set

$$S = \left\{ (\bullet_1, U) \xrightarrow{f} (\bullet_2, \top_{\bullet_2}) \,\middle|\, U \in L \right\}$$

is a sieve of the category $(\bullet \to \bullet \leftarrow \bullet) \rtimes \mathbb{L}$ on the object $(\bullet_2, \top_{\bullet_2})$ for which have that $\top_{\bullet_2} = \bigvee_S \exists_f U$ but also that $\top_{\bullet_3} \neq \bigvee g^*(S)$, as $g^*(S)$ is empty (here $\top_{\bullet_i}$ denotes the top element in $\mathbb{L}_{\bullet_i}$). Thus, $\mathbb{L}$ does not satisfy the relative Beck-Chevalley condition and therefore does not define an internal locale of $\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}$.

The subobject classifier $\Omega_{\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}}$ is, of course, an internal locale of the presheaf topos $\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}$. Recall (from [79, §I.4], say) that the subobject classifier $\Omega_{\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}}$, considered as a diagram in $\mathbf{Loc}_{\mathrm{open}}$, is given by

$$\mathbf{2} \xhookrightarrow{i_1} \mathbf{2}+\mathbf{2} \xhookleftarrow{i_2} \mathbf{2},$$

where $\mathbf{2}$ denotes the 2 element locale (i.e. the terminal locale) and $\mathbf{2}+\mathbf{2}$ is the coproduct in **Loc**. This is because there are two sieves, $\emptyset$ and $\{\mathrm{id}_1\}$, on $\bullet_1$, etc. Observe that the arrows i_1 and i_2 are disjoint open embeddings of locales, by which we mean that the following equations are satisfied, for all $V \in \mathbf{2}$,

$$i_1^{-1}\exists_{i_1} V = V,\ \ i_2^{-1}\exists_{i_2} V = \bot,\ \ i_2^{-1}\exists_{i_2} V = V,\ \ i_1^{-1}\exists_{i_2} V = \bot,$$

where $\bot$ represents the bottom element of $\mathbf{2}$. We show that this property characterises the internal locales of $\mathbf{Sets}^{(\bullet\to\bullet\leftarrow\bullet)^{\mathrm{op}}}$. We present this as a consequence of a wider theory regarding 'gluing' internal locales together.

Corollary II.18. *Let $\{ C_i \mid i \in I \}$ be a set of categories where, for each $i \in I$, C_i has a terminal object $\mathbf{1}_i$. Let $\mathcal{D}$ be the category obtained from the disjoint union $\coprod_{i\in I} C_i$ by freely adding a new terminal object $\mathbf{1}$. For each $i \in I$, we denote by $\mathbf{1}_i \xrightarrow{f_j} \mathbf{1}$ the newly added morphism connecting the respective terminal objects. A functor $\mathbb{L}\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ defines an internal locale of $\mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}}$ if and only if*

(i) for all $i \in I$,

$$\mathbb{L}|_{C_i} \colon C_i^{\mathrm{op}} \hookrightarrow \mathcal{D}^{\mathrm{op}} \xrightarrow{\mathbb{L}} \mathbf{Frm}_{\mathrm{open}}$$

is an internal locale of $\mathbf{Sets}^{C_i^{\mathrm{op}}}$,

(ii) and, for each pair $i, j \in I$ with $i \neq j$, the locale morphisms

$$\mathbb{L}_{\mathbf{1}_i} \xrightarrow{\mathbb{L}(f_i)} \mathbb{L}_{\mathbf{1}} \xleftarrow{\mathbb{L}(f_j)} \mathbb{L}_{\mathbf{1}_j}$$

are disjoint open embeddings of locales, by which we mean that, for all $V \in \mathbb{L}_{\mathbf{1}_i}$, $V' \in \mathbb{L}_{\mathbf{1}_j}$,

$$f_i^{-1}\exists_{f_i} V = V,\ \ f_j^{-1}\exists_{f_i} V = \bot_i,\ \ f_j^{-1}\exists_{f_j} V' = V',\ \ f_i^{-1}\exists_{f_j} V' = \bot_i,$$

where $\bot_i$ (respectively $\bot_j$) represents the bottom element of $\mathbb{L}_{\mathbf{1}_i}$ (resp. $\mathbb{L}_{\mathbf{1}_j}$).

Proof. For each object (d, V) of $\mathcal{D} \rtimes \mathbb{L}$, with d being an object of C_j say, a sieve S on (d, V) consists only of morphisms contained in $C_j \rtimes \mathbb{L}|_{C_j} \subseteq \mathcal{D} \rtimes \mathbb{L}$, and any arrow $e \xrightarrow{h} d$ of $\mathcal{D}$ is also contained in the subcategory $C_j \subseteq \mathcal{D}$. Therefore, we have that $h^{-1}(V) = \bigvee_{g\in h^*(S)} \exists_g U$ for each such V, S and h if and only if $\mathbb{L}|_{C_j}$ satisfies the relative Beck-Chevalley condition. We can thus limit our attention to the second criterion of the corollary and sieves on objects of the form $(\mathbf{1}, V) \in \mathcal{D} \rtimes \mathbb{L}$.

Suppose that $\mathbb{L}$ satisfies the relative Beck-Chevalley condition. For each $i \in I$ and $U \in \mathbb{L}_{\mathbf{1}_i}$, the principle sieve S generated by the arrow $(\mathbf{1}_i, U) \xrightarrow{f_i} (\mathbf{1}, \exists_{f_i} U)$ is $K_{\mathbb{L}}$-covering. Therefore

$$f_i^{-1}\exists_{f_i} U = \bigvee_{g\in f_i^*(S)} \exists_g W = U,$$

and so f_i is an open embedding. For each $j \in I$ with $i \neq j$, we have that

$$f_j^{-1}\exists_{f_i} U = \bigvee_{g\in f_j^*(S)} \exists_g W,$$

which, as $f_j^*(S)$ is empty, is equal to $\bot_i$ as required.

Conversely, suppose that $\mathbb{L}|_{C_i}$ is an internal locale of $\mathbf{Sets}^{C_i^{\mathrm{op}}}$, for each $i \in I$, and that $\mathbb{L}(f_i)$ and $\mathbb{L}(f_j)$ are disjoint open embeddings for each pair $i, j \in I$ with $i \neq j$. It remains to show that, if S is a sieve on $(\mathbf{1}, V)$ for which $V = \bigvee_{g \in S} \exists_g U$, then we have that

$$h^{-1}(V) = \bigvee_{g \in h^*(S)} \exists_{g'} U'$$

for any arrow $e \xrightarrow{h} \mathbf{1}$ of $\mathcal{D}$. It suffices to consider the case when h is the arrow $\mathbf{1}_j \xrightarrow{f_j} \mathbf{1}$, for some $j \in I$, and S is generated by arrows of the form $(\mathbf{1}_i, U) \xrightarrow{f_i} (\mathbf{1}, V)$. This is because any arrow h' can be factored as $e \to \mathbf{1}_j \xrightarrow{f_j} \mathbf{1}$ and any such sieve S can be rewritten as

$$\left\{ (c, U) \xrightarrow{g} (\mathbf{1}_i, \exists_g U) \xrightarrow{f_i} (\mathbf{1}, V) \,\middle|\, f_i \in T \right\}$$

where T generates a $K_{\mathbb{L}}$-covering sieve of the desired form. But now the thesis follows since $\mathbb{L}(f_i)$ and $\mathbb{L}(f_j)$ are disjoint open embeddings for each pair $i, j \in I$ with $i \neq j$. □

Example II.19. Using Corollary II.18, we are instantly able to recognise that a functor

$$\mathbb{L} \colon (\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}}$$

defines an internal locale of the topos $\mathbf{Sets}^{(\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}}}$ if and only if the diagram in **Loc**

$$\mathbb{L}_{\bullet_1} \xrightarrow{\ f\ } \mathbb{L}_{\bullet_2} \xleftarrow{\ g\ } \mathbb{L}_{\bullet_3}$$

is a pair of disjoint open embeddings, and thus confirm using Corollary II.18 that the constant functor $\mathbb{L} \colon (\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ considered in Example II.17 does not define an internal locale of $\mathbf{Sets}^{(\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}}}$.

More generally, if Γ is a *tree* (see [31, p. 26]), then the internal locales $\mathbf{Sets}^{\Gamma}$ are equivalently functors $\mathbb{L} \colon \Gamma^{\mathrm{op}} \to \mathbf{Loc}$ where, for each $x \in \Gamma$, the locale morphisms $\mathbb{L}_y \to \mathbb{L}_x$ corresponding to the *covers* of x (in the sense of [31, §1.14]) are disjoint open embeddings.

II.3.2 Internal locales for monoid actions

Although every topos has a site whose underlying category has pullbacks (e.g. the canonical site), there are many topoi which have a natural choice of site that lacks pullbacks. The classification of internal locales given in Section II.2 is most aptly applied when studying these topoi. An important example of such a topos is $\mathbf{B}G$, the topos of representations of a discrete group G on sets. This is the presheaf topos $\mathbf{Sets}^{G^{\mathrm{op}}}$, where the group G is viewed as a one-object category.

Therefore, applying Theorem II.2.1, we know that an internal locale of $\mathbf{Sets}^{G^{\mathrm{op}}}$ is a functor $\mathbb{L} \colon G^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ satisfying the relative Beck-Chevalley condition. But it is easily calculated that any action by G on a locale L by homeomorphisms, i.e. a group homomorphism $G \to \mathrm{Aut}_{\mathbf{Loc}}(L)$, also yields a functor

$$\mathbb{L} \colon G^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}}$$

that satisfies the relative Beck-Chevalley condition (this can be deduced as a corollary of the result for monoids below). Thus, by purely computational means we have recovered the correspondence between internal locales of $\mathbf{Sets}^{G^{\mathrm{op}}}$ and G-actions on locales that was also observed in [63, Example C2.5.8(d)].

However, for a monoid M, it is not true that any action by M on a locale L, i.e. a monoid homomorphism $M \to \mathrm{End}_{\mathbf{Loc}}(L)$, yields an internal locale of the topos of M-sets $\mathbf{Sets}^{M^{\mathrm{op}}}$. Nor will it suffice to restrict to *open* actions, those homomorphism that factor as $M \to \mathrm{End}_{\mathbf{Loc}_{\mathrm{open}}}(L) \subseteq \mathrm{End}_{\mathbf{Loc}}(L)$. Instead, an internal locale of $\mathbf{Sets}^{M^{\mathrm{op}}}$ must interact stably with respect to the set of *divisors* $\{\, k \in M \mid nk = m \,\}$, for $n, m \in M$, as described below.

Proposition II.20. *Let M be a monoid. An open action of M on a locale L constitutes an internal locale of* $\mathbf{Sets}^{M^{\mathrm{op}}}$ *if and only if, for each $U \in L$ and each pair $n, m \in M$,*

$$n^{-1}(\exists_m U) = \bigvee_{\substack{k \in M \\ nk = m}} \exists_k U.$$

Proof. We must show that the functor $\mathbb{L} \colon M^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ induced by the open action of M on L satisfies the relative Beck-Chevalley condition if and only if, for each $U \in L$ and each pair $n, m \in M$,

$$n^{-1}(\exists_m U) = \bigvee_{\substack{k \in M \\ nk = m}} \exists_k U.$$

Assuming the relative Beck-Chevalley condition, the $K_{\mathbb{L}}$-covering sieve R generated by the single arrow $(*, U) \xrightarrow{m} (*, \exists_m U)$ must be stable under the map

$$(*, n^{-1}\exists_m U) \xrightarrow{\ n\ } (*, U)$$

We readily calculate that

$$n^*(R) = \left\{ (*, V) \xrightarrow{k} (*, n^{-1}\exists_m U) \,\middle|\, nk = m \text{ and } V \leqslant k^{-1}n^{-1}\exists_m U \right\}.$$

Hence, we have that

$$n^{-1}(\exists_m U) = \bigvee_{k \in n^*(R)} \exists_k V.$$

By the inequality

$$V \leqslant k^{-1}n^{-1}\exists_m U = k^{-1}n^{-1}\exists_n \exists_k U \leqslant U,$$

we deduce that $\exists_k V \leqslant \exists_k U$. Simultaneously, the equality $\exists_n \exists_k U = \exists_m U$ implies that $\exists_k U \leqslant n^{-1}(\exists_m U)$. Combining the two inequalities, we conclude that

$$n^{-1}(\exists_m U) = \bigvee_{k \in n^*(R)} \exists_k V = \bigvee_{\substack{k \in M \\ nk = m}} \exists_k U$$

as required.

For the converse, let S be a sieve in $M \rtimes \mathbb{L}$ on the object $(*, V)$ for which $V = \bigvee_{m \in S} \exists_f U$. Then we calculate that

$$\begin{aligned} n^{-1}(V) &= \bigvee_{m \in S} n^{-1}\exists_m U, \\ &= \bigvee_{m \in S} \bigvee_{\substack{k \in M \\ nk = m}} \exists_k U. \end{aligned}$$

We need only finally note that

$$n^*(S) = \left\{ (*, U) \xrightarrow{k} (*, V) \,\middle|\, \exists m \in S, nk = m \right\}$$

to deduce the result. □

Example II.21. For the monoid $(\mathbb{N}, +, 0)$, Proposition II.20 recovers the characterisation of *difference locales* given in [116, Propositoin 13.10].

II.4 Internal locale morphisms

In this section we begin our study the morphisms of internal locales and their properties. We aim to provide a parallel to the treatment of locale morphisms and the geometric morphisms between localic topoi that is found in [79, §IX]. Therein it is shown that, given two locales X, Y (of **Sets**), there is an equivalence

$$\mathbf{Loc}(X, Y) \simeq \mathbf{Geom}(\mathbf{Sh}(X), \mathbf{Sh}(Y)) \tag{II.ii}$$

between the category of locale morphisms $X \to Y$ and the category of geometric morphisms $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The morphisms of internal locales were first characterised in [68, §VI.2].

Definition II.22 (Proposition VI.2.1 [68])**.** Given a pair $\mathbb{L}_1, \mathbb{L}_2 \colon C^{\mathrm{op}} \rightrightarrows \mathbf{Frm}_{\mathrm{open}}$ of internal locales of the topos $\mathbf{Sh}(C, J)$, an *internal locale morphism* $\mathfrak{f} \colon \mathbb{L}_1 \to \mathbb{L}_2$ consists of a frame homomorphism $\mathfrak{f}_c^{-1} \colon \mathbb{L}_2(c) \to \mathbb{L}_1(c)$, such that, for each morphism $c \xrightarrow{g} d$ of C, the diagram

$$\begin{tikzcd}
\mathbb{L}_2(d) \arrow[r, shift right, "\mathbb{L}_2(g)"'] \arrow[d, "\mathfrak{f}_d^{-1}"] & \mathbb{L}_2(c) \arrow[l, shift right, "\exists_{\mathbb{L}_2(g)}"'] \arrow[d, "\mathfrak{f}_c^{-1}"] \\
\mathbb{L}_1(d) \arrow[r, shift right, "\mathbb{L}_1(g)"'] & \mathbb{L}_1(c) \arrow[l, shift right, "\exists_{\mathbb{L}_1(g)}"']
\end{tikzcd}$$

is a *morphism of adjunctions*: i.e., the equations

$$\mathbb{L}_1(g) \circ \mathfrak{f}_d^{-1} = \mathfrak{f}_c^{-1} \circ \mathbb{L}_2(g) \text{ and } \exists_{\mathbb{L}_1(g)} \circ \mathfrak{f}_c^{-1} = \mathfrak{f}_d^{-1} \circ \exists_{\mathbb{L}_2(g)}$$

are both satisfied.

Our first task is to extend the equivalence (II.ii) between internal locale morphisms and geometric morphisms for set-based locales to the internal setting, as demonstrated concretely in [24, §4]. We construct a bijective correspondence between

(i) the internal locale morphisms $\mathfrak{f} \colon \mathbb{L}_1 \to \mathbb{L}_2$;

(ii) the morphisms of relative sites

$$\left[(C \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) \xrightarrow{\pi_{\mathbb{L}_2}} (C, J)\right] \longrightarrow \left[(C \rtimes \mathbb{L}_1, K_{\mathbb{L}_1}) \xrightarrow{\pi_{\mathbb{L}_1}} (C, J)\right],$$

i.e. the morphisms of fibrations

$$\begin{tikzcd} \mathcal{C} \rtimes \mathbb{L}_2 \arrow[rr, "\mathrm{id}_{\mathcal{C}} \rtimes \mathfrak{f}^{-1}"] \arrow[dr, "\pi_{\mathbb{L}_2}"'] & & \mathcal{C} \rtimes \mathbb{L}_1 \arrow[dl, "\pi_{\mathbb{L}_1}"] \\ & \mathcal{C}, & \end{tikzcd}$$

where $\mathfrak{f}^{-1} \colon \mathbb{L}_2 \Rightarrow \mathbb{L}_1$ denotes a natural transformation, for which the induced functor $\mathrm{id}_{\mathcal{C}} \rtimes \mathfrak{f}^{-1}$ yields a morphisms of sites $\mathrm{id}_{\mathcal{C}} \rtimes \mathfrak{f}^{-1} \colon (\mathcal{C} \rtimes \mathbb{L}_1, K_{\mathbb{L}_1}) \to (\mathcal{C} \rtimes \mathbb{L}_2, K_{\mathbb{L}_2})$ – for notational convenience, we denote the functor $\mathrm{id}_{\mathcal{C}} \rtimes \mathfrak{f}^{-1}$ by $\check{\mathfrak{f}}$;

(iii) finally, the geometric morphisms $f \colon \mathbf{Sh}(\mathbb{L}_1) \to \mathbf{Sh}(\mathbb{L}_2)$ for which the triangle

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{L}_1) \arrow[rr, "f"] \arrow[dr, "C_{\pi_{\mathbb{L}_1}}"'] & & \mathbf{Sh}(\mathbb{L}_2) \arrow[dl, "C_{\pi_{\mathbb{L}_2}}"] \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{tikzcd} \tag{II.iii}$$

commutes.

Thereby, we will recover the biequivalence

$$\mathbf{Loc}\,(\mathbf{Sh}(\mathcal{C}, J)) \simeq \mathfrak{Loc}/\mathbf{Sh}(\mathcal{C}, J) \tag{II.iv}$$

(as seen in [24, Corollary 3.5]).

(i) Here, $\mathbf{Loc}\,(\mathbf{Sh}(\mathcal{C}, J))$ denotes the bicategory of internal locales of $\mathbf{Sh}(\mathcal{C}, J)$, their internal locale morphisms and natural transformations between these.

(ii) By $\mathfrak{Loc}/\mathbf{Sh}(\mathcal{C}, J)$ we denote the bicategory whose objects are localic geometric morphisms $f \colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$, whose 1-cells are commuting geometric morphisms

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{E}' \arrow[dl, "f'"] \\ & \mathbf{Sh}(\mathcal{C}, J), & \end{tikzcd}$$

(the geometric morphism g is also localic by [59, Lemma 1.1(ii)]) and whose 2-cells are the commuting 2-cells of geometric morphisms

$$\begin{tikzcd} \mathcal{E} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "f"'] & & \mathcal{E}' \arrow[dl, "f'"] \\ & \mathbf{Sh}(\mathcal{C}, J). & \end{tikzcd}$$

Having related internal locale morphisms and geometric morphisms, we turn to a study of their properties. In Proposition II.28, we will extend, to the to internal setting, the result [79, Proposition IX.5.5(i)], which states that a locale morphism $f \colon L \to K$ is an surjective locale morphism if and only if the induced geometric morphism $\mathbf{Sh}(f) \colon \mathbf{Sh}(L) \to \mathbf{Sh}(K)$ between localic topoi is surjective. Further properties of internal locale morphisms shall be studied in Section II.5 and Section II.6.

Internal locale morphisms and geometric morphisms.

Proposition II.23. *Let* $\mathbb{L}_1, \mathbb{L}_2 \colon C^{\mathrm{op}} \rightrightarrows \mathbf{Frm}_{\mathrm{open}}$ *be internal locales of* $\mathbf{Sh}(C, J)$*. There is an equivalence of categories*

$$\mathbf{Loc}(\mathbf{Sh}(C, J))(\mathbb{L}_1, \mathbb{L}_2) \simeq \mathfrak{Loc}/\mathbf{Sh}(C, J)(\mathbf{Sh}(\mathbb{L}_1), \mathbf{Sh}(\mathbb{L}_2)).$$

Proof. By Corollary I.23, Corollary I.25 and the isomorphism $\mathbb{L}_1 \cong \mathrm{Sub}_{\mathbf{Sh}(\mathbb{L}_1)}(C^*_{\pi_{\mathbb{L}_1}} \ell_C -)$, there is an equivalence of categories

$$\begin{aligned}
&\mathfrak{Loc}/\mathbf{Sh}(C, J)(\mathbf{Sh}(\mathbb{L}_1), \mathbf{Sh}(\mathbb{L}_2)) \\
\simeq\, &\mathbf{Topos}/\mathrm{id}_{\mathcal{E}} \left(\begin{array}{ccc} \mathbf{Sh}(\mathbb{L}_1) & & \mathbf{Sh}(\mathbb{L}_2) \\ \downarrow{\scriptstyle C_{\pi_{\mathbb{L}_1}}} & , & \downarrow{\scriptstyle C_{\pi_{\mathbb{L}_2}}} \\ \mathcal{E} & & \mathcal{E} \end{array} \right), \\
\simeq\, &\mathbf{RelMorph}/\ell_C \left(\begin{array}{ccc} (C \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) & & (\mathcal{E} \rtimes \mathrm{Sub}_{\mathbf{Sh}(\mathbb{L}_1)}(C^*_{\pi_{\mathbb{L}_1}} -), \tilde{J}_{\mathrm{can}}) \\ \downarrow{\scriptstyle \pi_{\mathbb{L}_2}} & , & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ (C, J) & & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right), \\
\simeq\, &\mathbf{RelMorph}/\mathrm{id}_C \left(\begin{array}{ccc} (C \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) & & (C \rtimes \mathrm{Sub}_{\mathbf{Sh}(\mathbb{L}_1)}(C^*_{\pi_{\mathbb{L}_1}} \ell_C -), \tilde{J}_{\mathrm{can}}) \\ \downarrow{\scriptstyle \pi_{\mathbb{L}_2}} & , & \downarrow{\scriptstyle \pi_{\mathrm{Sub}_{\mathbf{Sh}(\mathbb{L}_1)}(C^*_{\pi_{\mathbb{L}_1}} \ell_C -)}} \\ (C, J) & & (C, J) \end{array} \right) \\
\simeq\, &\mathbf{RelMorph}/\mathrm{id}_C \left(\begin{array}{ccc} (C \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) & & (C \rtimes \mathbb{L}_1, K_{\mathbb{L}_1}) \\ \downarrow{\scriptstyle \pi_{\mathbb{L}_2}} & , & \downarrow{\scriptstyle \pi_{\mathbb{L}_1}} \\ (C, J) & & (C, J) \end{array} \right)
\end{aligned}$$

By a restriction of the equivalence in Proposition I.28, the latter is also equivalent to the category

$$\mathbf{RelMorph}_{\mathrm{cart}}/\mathrm{id}_C\big((C, J, \mathbb{L}_2, K_{\mathbb{L}_2}), (C, J, \mathbb{L}_1, K_{\mathbb{L}_1})\big),$$

the category whose objects are morphisms of relative sites of the form

$$\left(\check{f}, \mathrm{id}_C\right) \colon \left[(C \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) \xrightarrow{\pi_{\mathbb{L}_2}} (C, J)\right] \longrightarrow \left[(C \rtimes \mathbb{L}_1, K_{\mathbb{L}_1}) \xrightarrow{\pi_{\mathbb{L}_1}} (C, J)\right],$$

and whose arrows are natural transformations between these, or equivalently the category:

(i) whose objects are natural transformations $\mathfrak{f}^{-1} \colon \mathbb{L}_2 \Rightarrow \mathbb{L}_1$, where each component $\mathfrak{f}_c^{-1} \colon \mathbb{L}_2(c) \to \mathbb{L}_1(c)$ preserves finite limits (ie. meets) and for which the induced natural transformation

$$\check{\mathfrak{f}} \colon C \rtimes \mathbb{L}_2 \longrightarrow C \rtimes \mathbb{L}_1$$

sends $K_{\mathbb{L}_2}$-covers to $K_{\mathbb{L}_1}$-covers,

(ii) and whose arrows are modifications between these.

Hence, to establish the equivalence

$$\mathbf{Loc}(\mathbf{Sh}(C, J))(\mathbb{L}_1, \mathbb{L}_2) \simeq \mathfrak{Loc}/\mathbf{Sh}(C, J)(\mathbf{Sh}(\mathbb{L}_1), \mathbf{Sh}(\mathbb{L}_2)),$$

it remains only to show that the induced functor $\breve{\mathfrak{f}}$ of a pointwise cartesian natural transformation $\mathfrak{f}^{-1}\colon \mathbb{L}_2 \Rightarrow \mathbb{L}_1$ is cover preserving if and only if $\mathfrak{f}^{-1}$ defines an internal locale morphism.

To that end, it suffices to consider the two generating species of $K_{\mathbb{L}_2}$-covering families identified in Proposition II.14. Let

$$\left\{ (c, U) \xrightarrow{g} (c, \exists_{\mathbb{L}_2(g)} U) \right\}$$

be a $K_{\mathbb{L}_2}$-covering family of species (A). The family

$$\breve{\mathfrak{f}}\left(\left\{ (c, U) \xrightarrow{g} (c, \exists_{\mathbb{L}_2(g)} U) \right\}\right) = \left\{ (c, \mathfrak{f}_c^{-1}(U)) \xrightarrow{g} (c, \mathfrak{f}_d^{-1}(\exists_{\mathbb{L}_2(g)} U)) \right\}$$

is $K_{\mathbb{L}_1}$-covering if and only if $\mathfrak{f}_d^{-1}(\exists_{\mathbb{L}_2(g)} U) = \exists_{\mathbb{L}_1(g)} \mathfrak{f}_c^{-1}(U)$. Let

$$\left\{ (c, U_i) \xrightarrow{\mathrm{id}_c} (c, \bigvee_{i \in I} U_i) \,\middle|\, i \in I \right\}$$

be a $K_{\mathbb{L}_2}$-covering family of species (B). The family

$$\breve{\mathfrak{f}}\left(\left\{ (c, U_i) \xrightarrow{\mathrm{id}_c} \left(c, \bigvee_{i \in I} U_i\right) \,\middle|\, i \in I \right\}\right) = \left\{ (c, \mathfrak{f}_c^{-1}(U_i)) \xrightarrow{\mathrm{id}_c} \left(c, \mathfrak{f}_c^{-1}\left(\bigvee_{i \in I} U_i\right)\right) \,\middle|\, i \in I \right\}$$

is $K_{\mathbb{L}_1}$-covering if and only if $\mathfrak{f}_c^{-1}$ preserves joins, and hence is a frame homomorphism, thus completing the proof. □

Theorem II.24 (Corollary 3.5 [24]). *There is a biequivalence*

$$\mathbf{Loc}(\mathbf{Sh}(C, J)) \simeq \mathfrak{Loc}/\mathbf{Sh}(C, J).$$

Proof. By Proposition II.23, the action on objects that sends a localic geometric morphism $f\colon \mathcal{E} \to \mathbf{Sh}(C, J)$ to the internal locale $f_*(\Omega_{\mathcal{E}})$ can be extended to a bifunctor $\mathfrak{L}\colon \mathfrak{Loc}/\mathbf{Sh}(C, J) \to \mathbf{Loc}\,(\mathbf{Sh}(C, J))$. Similarly, the action on objects that sends an internal locale $\mathbb{L}$ to the localic geometric morphism

$$C_{\pi_{\mathbb{L}}}\colon \mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}}) \longrightarrow \mathbf{Sh}(C, J)$$

also extends to a bifunctor $\mathfrak{T}\colon \mathbf{Loc}\,(\mathbf{Sh}(C, J)) \to \mathfrak{Loc}/\mathbf{Sh}(C, J)$.

By Proposition II.23, the isomorphism $\mathbb{L} \cong C_{\pi_{\mathbb{L}*}}\left(\Omega_{\mathbf{Sh}(\mathbb{L})}\right)$ and the isomorphism $f \simeq C_{\pi_{f_*(\Omega_{\mathcal{F}})}}$, the bifunctors $\mathfrak{L}$ and $\mathfrak{T}$ are mutually inverse. □

Notation II.25. Given an internal locale morphism $\mathfrak{f}\colon \mathbb{L}_1 \to \mathbb{L}_2$, we use

$$\mathbf{Sh}(\mathfrak{f})\colon \mathbf{Sh}(\mathbb{L}_1) \longrightarrow \mathbf{Sh}(\mathbb{L}_2)$$

to denote the corresponding localic geometric morphism.

Corollary II.26. *The subobject classifier $\Omega_{\mathcal{E}}$ of a topos is the terminal object of* $\mathbf{Loc}(\mathcal{E})$.

Proof. The identity $\mathrm{id}_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E}$ is the terminal object of $\mathfrak{Loc}/\mathcal{E}$. □

II.4.1 Surjective internal locale morphisms

We now turn to characterising some properties of the geometric morphisms induced by internal locale morphisms. Recall that a locale morphism $f\colon L \to K$ is a *surjection* if the corresponding frame homomorphism $f^{-1}\colon K \to L$ is injective. Recall also that a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is a *surjection* if the inverse image functor $f^*\colon \mathcal{E} \to \mathcal{F}$ is faithful. In [79, Proposition X.5.5(i)], it is shown that a locale morphism $f\colon L \to K$ is surjective if and only if the corresponding geometric morphism $\mathbf{Sh}(f)\colon \mathbf{Sh}(L) \to \mathbf{Sh}(K)$ is surjective. We extend this to the internal setting, and show that surjections of internal locales can be characterised 'pointwise'.

Definition II.27. Let $\mathfrak{f}\colon \mathbb{L}_1 \to \mathbb{L}_2$ be an internal locale morphism of $\mathbf{Sh}(\mathcal{C}, J)$. We say that $\mathfrak{f}$ is a *surjective internal locale morphism* if $\mathfrak{f}_c^{-1}\colon \mathbb{L}_2(c) \to \mathbb{L}_1(c)$ is injective, for each object $c \in \mathcal{C}$.

Proposition II.28. *Let* $\mathfrak{f}\colon \mathbb{L}_1 \to \mathbb{L}_2$ *be an internal locale morphism of* $\mathbf{Sh}(\mathcal{C}, J)$*. The following are equivalent:*

(i) the geometric morphism $\mathbf{Sh}(\mathfrak{f})$ *is a surjective,*

(ii) $\mathfrak{f}$ *is a surjective internal locale morphism.*

Proof. By [23, Theorem 6.3], the geometric morphism $\mathbf{Sh}(\mathfrak{f})$ is surjective if and only if the corresponding morphism of sites

$$\check{\mathfrak{f}}\colon (\mathcal{C} \rtimes \mathbb{L}_2, K_{\mathbb{L}_2}) \longrightarrow (\mathcal{C} \rtimes \mathbb{L}_1, K_{\mathbb{L}_1})$$

is cover reflecting. Suppose that each $\mathfrak{f}_d^{-1}$ is injective. Let S be sieve of $\mathcal{C} \rtimes \mathbb{L}_2$ on (d, V) such that $\check{\mathfrak{f}}(S)$ is $K_{\mathbb{L}_1}$-covering, i.e. $\mathfrak{f}_d^{-1}(V) = \bigvee_{g \in S} \exists_{\mathbb{L}_1(g)} \mathfrak{f}_c^{-1}(U)$. We have that

$$\begin{aligned}
\mathfrak{f}_d^{-1}(V) &= \bigvee_{g \in S} \exists_{\mathbb{L}_1(g)} \mathfrak{f}_c^{-1}(U), \\
&= \bigvee_{g \in S} \mathfrak{f}_d^{-1} \exists_{\mathbb{L}_2(g)} U, \\
&= \mathfrak{f}_d^{-1}\left(\bigvee_{g \in S} \exists_{\mathbb{L}_2(g)} U \right).
\end{aligned}$$

Thus, since $\mathfrak{f}_d^{-1}$ is injective $V = \bigvee_{g \in S} \exists_{\mathbb{L}_2(g)} U$ and so S is $K_{\mathbb{L}_2}$-covering.

Conversely, if $\check{\mathfrak{f}}$ is cover reflecting and $\mathfrak{f}_c^{-1}(U) = \mathfrak{f}_c^{-1}(V)$ for a pair of elements $U, V \in \mathbb{L}_2(c)$, then $\check{\mathfrak{f}}$ reflects the maximal cover. Hence, we conclude that $U = V$. □

II.5 Internal embeddings and nuclei

This section is dedicated to the study of internal locale embeddings. Their study is continued in Section II.6. Recall that a locale morphism $f\colon K \to L$ is said to be an *embedding* if the corresponding frame homomorphism $f^{-1}\colon L \to K$ is surjective – or equivalently if the right adjoint $f_*\colon K \to L$ is injective (see [79, Lemma IX.4.2]). Just as with surjective internal locale morphisms, we define internal locale embeddings as the 'pointwise' generalisation.

Definition II.29. Let $\mathfrak{f} \colon \mathbb{L}_1 \to \mathbb{L}_2$ be an internal locale morphism of the topos $\mathbf{Sh}(C, J)$. We say that $\mathfrak{f}$ is an *internal locale embedding* if $\mathfrak{f}_c^{-1} \colon \mathbb{L}_2(c) \to \mathbb{L}_1(c)$ is surjective, for each object $c \in C$. We will also refer to $\mathbb{L}_1$ as an *internal sublocale* of $\mathbb{L}_2$ and $\mathfrak{f}$ as the *inclusion* of this internal sublocale.

Recall also that a geometric morphism $f \colon \mathcal{F} \to \mathcal{E}$ is said to be a *geometric embedding* (and $\mathcal{F}$ a *subtopos* of $\mathcal{E}$) if the direct image functor f_* is full and faithful. By [79, Proposition IX.5.4], geometric embeddings generalise embeddings of sublocales in the sense that, given a locale morphism $f \colon K \to L$, the induced geometric morphism

$$\mathbf{Sh}(f) \colon \mathbf{Sh}(K) \longrightarrow \mathbf{Sh}(L)$$

between the topoi of sheaves is a geometric embedding if and only if f is an embedding of locales.

The aim of this section is to prove an analogous result for embeddings of internal locales: that, given a morphism of internal locales $\mathfrak{f} \colon \mathbb{L}' \to \mathbb{L}$ of $\mathbf{Sh}(C, J)$, the geometric morphism $\mathbf{Sh}(\mathfrak{f}) \colon \mathbf{Sh}(\mathbb{L}') \to \mathbf{Sh}(\mathbb{L})$ is an embedding if and only if $\mathfrak{f}$ is an internal locale embedding. To this end, we develop a study of *internal nuclei*. These are the internal generalisations of the nuclei on a locale, and will appear reminiscent of *Lawvere-Tierney topologies* (a similarity that will be made concrete in Theorem II.34). Nuclei are a useful tool when studying sublocales since many properties of sublocales are more readily proven using nuclei than directly. In particular, that the sublocales of a locale L form a co-frame is often proved via nuclei, as discussed in Section II.6 below.

Overview. We proceed as follows.

- In Section II.5.1, the notion of an internal nucleus on an internal locale $\mathbb{L}$ is introduced and it is shown that internal nuclei correspond bijectively with internal sublocales of $\mathbb{L}$.
- We show in Section II.5.2 that internal nuclei on $\mathbb{L}$, and thus by extension internal sublocales of $\mathbb{L}$, correspond bijectively with Lawvere-Tierney topologies on $\Omega_{\mathbf{Sh}(\mathbb{L})}$, and hence subtopoi of $\mathbf{Sh}(\mathbb{L})$.
- Finally, in Section II.5.3, we conclude that the surjection-inclusion factorisation of a localic geometric morphism is calculated 'pointwise'.

II.5.1 Internal nuclei

Recall from [60, §II.2] that a nucleus on a locale L is a function $j \colon L \to L$ satisfying, for all $x, y \in L$,

$$x \leqslant j(x), \quad j(j(x)) \leqslant j(x), \quad j(x \wedge y) = j(x) \wedge j(y).$$

These properties are referred to as j being, respectively, *inflationary*, *idempotent*, and *meet-preserving*. Any function satisfying these properties must also be monotone.

It is well-known (see [60, Theorem II.2.3]) that there is a bijective correspondence between nuclei on L and sublocales of L. In one direction, the nucleus associated to a sublocale $f \colon K \rightarrowtail L$ is given by the function $f_* f^{-1} \colon L \to L$ (here f_* denotes the right adjoint to f^{-1}, see Notation II.2). Conversely, given a nucleus $j \colon L \to L$, the image of j as a subset of L, which we denote by L^j, can be given the structure of a frame.

The meets are computed as they are in L while the join of a subset $\{ U_i \mid i \in I \} \subseteq L^j$ is computed as $j(\bigvee_{i\in I} U_i)$, where $\bigvee_{i\in I} U_i$ is the join in L. It is then clear that $j\colon L \to L^j$ constitutes a surjective frame homomorphism, and hence the inclusion of a sublocale (see [60, Lemma II.2.2] or [79, Proposition IX.4.3]).

Definition II.30. Let $\mathbb{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ be an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$. An *internal nucleus* is a natural transformation $j\colon \mathbb{L} \to \mathbb{L}$ (as a functor into **Sets**) such that each component $j_c\colon \mathbb{L}_c \to \mathbb{L}_c$, for $c \in \mathcal{C}$, is a nucleus on the locale $\mathbb{L}_c$.

When the subobject classifier $\Omega_{\mathbf{Sh}(\mathcal{C},J)}$ of $\mathbf{Sh}(\mathcal{C}, J)$ is considered as an internal locale, the definition of an internal nucleus $j\colon \Omega_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_{\mathbf{Sh}(\mathcal{C},J)}$ coincides with that of a *Lawvere-Tierney topology* (see [63, Definition A4.4.1]). For a localic geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$, we observe below that internal nuclei on $f_*(\Omega_{\mathcal{F}})$ correspond bijectively with Lawvere-Tierney topologies on $\Omega_{\mathcal{F}}$.

First, we establish a bijective correspondence between internal nuclei and internal sublocales that generalises the bijective correspondence for locales (see [60, Theorem II.2.3]).

Lemma II.31. *Let $j\colon L \to L$ be a nucleus. For each subset $\{ U_i \mid i \in I \} \subseteq L$, we have that*

$$j\left(\bigvee_{i\in I} U_i\right) = j\left(\bigvee_{i\in I} jU_i\right).$$

Proof. The first inequality $j(\bigvee_{i\in I} U_i) \leqslant j(\bigvee_{i\in I} jU_i)$ is a consequence of j being inflationary as $U_i \leqslant jU_i$ for each $i \in I$. The converse inequality is achieved by applying j to both sides of the canonical inequality

$$\bigvee_{i\in I} jU_i \leqslant j\left(\bigvee_{i\in I} U_i\right).$$

□

Proposition II.32. *Each internal nucleus j on an internal locale $\mathbb{L}$ of $\mathbf{Sh}(\mathcal{C}, J)$ defines an embedding of internal locales $\mathbb{L}^j \hookrightarrow \mathbb{L}$.*

Proof. By the above discussion, for each object c of $\mathcal{C}$, the nucleus

$$j_c\colon \mathbb{L}_c \longrightarrow \mathbb{L}_c$$

induces a sublocale $\mathbb{L}_c^j$ of $\mathbb{L}_c$. As j is a natural transformation, for each arrow $c \xrightarrow{g} d$ of $\mathcal{C}$, $g^{-1}\colon \mathbb{L}_d \to \mathbb{L}_c$ restricts to a function $g^{-1}\colon \mathbb{L}_d^j \to \mathbb{L}_c^j$ which, by the definition of meets and joins in $\mathbb{L}_d^j$ and $\mathbb{L}_c^j$, can easily be shown to be a frame homomorphism. We must therefore show that each $g^{-1}\colon \mathbb{L}_d^j \to \mathbb{L}_c^j$ is also open.

A left adjoint is given by $j_d \exists_{\mathbb{L}(g)}$ since, for each $U \in \mathbb{L}_c^j$ and $V \in \mathbb{L}_d^j$,

$$j_d \exists_{\mathbb{L}(g)} U \leqslant V = j_d(V) \iff \exists_{\mathbb{L}(g)} U \leqslant V \iff U \leqslant g^{-1}(V),$$

and furthermore the Frobenius condition is satisfied:

$$j_d \exists_{\mathbb{L}(g)} U \wedge V = j_d \exists_{\mathbb{L}(g)} U \wedge j_d V = j_d((\exists_{\mathbb{L}(g)} U) \wedge V) = j_d \exists_{\mathbb{L}(g)} (U \wedge g^{-1}(V)).$$

We thus conclude that each internal nucleus j induces a functor

$$\mathbb{L}^j \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}.$$

Moreover, we observe that the square

$$\begin{tikzcd} \mathbb{L}_c \arrow[r, "\exists_g"] \arrow[d, "j_c"'] & \mathbb{L}_d \arrow[d, "j_d"] \\ \mathbb{L}_c^{j_c} \arrow[r, "j_d \exists_g"] & \mathbb{L}_d^{j_d} \end{tikzcd}$$

commutes for each $c \xrightarrow{g} d \in \mathcal{C}$. For each $U \in \mathbb{L}_c$, we have that $U \leqslant j_c(U)$ and so

$$j_d \exists_g U \leqslant j_d \exists_g j_c(U).$$

For the converse inequality, as $U \leqslant g^{-1}\exists_g U$, it follows that

$$\begin{aligned} U \leqslant g^{-1}\exists_g U &\implies j_d(U) \leqslant j_d g^{-1}\exists_g(U), \\ &\implies j_d(U) \leqslant g^{-1} j_c \exists_g(U), \\ &\implies \exists_g j_d(U) \leqslant j_c \exists_g(U), \\ &\implies j_c \exists_g j_d(U) \leqslant j_c \exists_g(U). \end{aligned}$$

Therefore, we have a natural transformation $j \colon \mathbb{L} \to \mathbb{L}^j$, where each component is a surjective frame homomorphism and, moreover, j commutes with the respective left adjoints, i.e. $j_d \exists_g j_c = j_d \exists_g$ for each arrow $d \xrightarrow{g} c$ of $\mathcal{C}$. Hence, j would define an embedding of internal locales if $\mathbb{L}^j$ were also an internal locale of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

To show that $\mathbb{L}^j$ is an internal locale, it remains only to show that the functor $\mathbb{L}^j$ satisfies the relative Beck-Chevalley condition. Let S be a sieve on $(d, V) \in \mathcal{C} \rtimes \mathbb{L}^j$ such that

$$V = j_d\left(\bigvee_{g \in S} j_d \exists_{\mathbb{L}(g)} U\right),$$

which, by Lemma II.31, is equal to $j_d\left(\bigvee_{g\in S} \exists_{\mathbb{L}(g)} U\right)$, and let $e \xrightarrow{h} d$ be an arrow of $\mathcal{C}$. For notational convenience, let W denote $\bigvee_{g\in S} \exists_{\mathbb{L}(g)} U$. Since $\mathbb{L}$ is an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$,

$$h^{-1}(W) = \bigvee_{g \in h^*(S)} \exists_{\mathbb{L}(g)} U.$$

Thus, by Lemma II.31, we have the desired equality

$$h^{-1}(V) = h^{-1}(j_d(W)) = j_e(h^{-1}(W)) = j_e\left(\bigvee_{g\in h^*(S)} \exists_{\mathbb{L}(g)} U\right) = j_e\left(\bigvee_{g \in h^*(S)} j_e \exists_{\mathbb{L}(g)} U\right),$$

and therefore $\mathbb{L}^j$ is an internal locale of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Since $\mathbf{Sh}(\mathbb{L}^j) \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ factors as

$$\mathbf{Sh}(\mathbb{L}^j) \longrightarrow \mathbf{Sh}(\mathbb{L}) \longrightarrow \mathbf{Sh}(\mathcal{C}, J) \rightarrowtail \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}},$$

we conclude that $\mathbb{L}^j$ is an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$ as well by Lemma II.16. $\square$

Corollary II.33. *Let $\mathbb{L}\colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ be an internal locale of $\mathbf{Sh}(C, J)$. There is a bijective correspondence between internal sublocales of $\mathbb{L}$ and internal nuclei on $\mathbb{L}$.*

Proof. By the theory of standard locales, there is a bijective correspondence between collections of nuclei

$$\{\, j_c \colon \mathbb{L}_c \to \mathbb{L}_c \mid c \in C \,\}$$

and collections of sublocales

$$\{\, f_c \colon \mathbb{L}'_c \rightarrowtail \mathbb{L}_c \mid c \in C \,\},$$

where both are indexed by the objects of C. Our bijection will be a restriction of this correspondence.

We have already seen in Proposition II.32 that if the collection

$$\{\, j_c \colon \mathbb{L}_c \to \mathbb{L}_c \mid c \in C \,\}$$

of nuclei is natural in c, i.e. it defines an internal nucleus, then the corresponding collection of sublocales yields an internal sublocale embedding. It remains to show the other direction: that if

$$\{\, \mathfrak{f}_c \colon \mathbb{L}'_c \rightarrowtail \mathbb{L}_c \mid c \in C \,\}$$

are the components of an internal sublocale embedding, then the corresponding collection of nuclei is natural.

Let $\mathfrak{f}\colon \mathbb{L}' \to \mathbb{L}$ be an embedding of an internal sublocale. Since each component $\mathfrak{f}_c^{-1}\colon \mathbb{L}'_c \to \mathbb{L}_c$ is surjective, it induces a nucleus $\mathfrak{f}_{*c}\mathfrak{f}_c^{-1}\colon \mathbb{L}_c \to \mathbb{L}_C$, for each object c of C. We wish to show that, for each arrow $c \xrightarrow{g} d$ of C, the square

$$\begin{tikzcd}
\mathbb{L}_d \arrow[r, "g^{-1}"] \arrow[d, "\mathfrak{f}_{*d}\mathfrak{f}_d^{-1}"'] & \mathbb{L}_c \arrow[d, "\mathfrak{f}_{*c}\mathfrak{f}_c^{-1}"] \\
\mathbb{L}_d \arrow[r, "g^{-1}"] & \mathbb{L}_c
\end{tikzcd}$$

commutes. Since the square

$$\begin{tikzcd}
\mathbb{L}_d \arrow[r, shift right, "g^{-1}"'] \arrow[d, "\mathfrak{f}_d^{-1}"'] & \mathbb{L}_c \arrow[l, shift right, "\exists_g"'] \arrow[d, "\mathfrak{f}_c^{-1}"] \\
\mathbb{L}_d \arrow[r, shift right, "g^{-1}"'] & \mathbb{L}_c, \arrow[l, shift right, "\exists_g"']
\end{tikzcd}$$

is a morphism of adjunctions, taking the respective right adjoints also yields a morphism of adjunctions

$$\begin{tikzcd}
\mathbb{L}_d \arrow[r, shift left, "g^{-1}"] & \mathbb{L}_c \arrow[l, shift left, "g_*"] \\
\mathbb{L}_d \arrow[r, shift left, "g^{-1}"] \arrow[u, "\mathfrak{f}_{*d}"] & \mathbb{L}_c. \arrow[l, shift left, "g_*"] \arrow[u, "\mathfrak{f}_{*c}"']
\end{tikzcd}$$

Hence we have the desired equality

$$\mathfrak{f}_{*c}\mathfrak{f}_c^{-1}g^{-1} = \mathfrak{f}_{*c}g^{-1}\mathfrak{f}_d^{-1} = g^{-1}\mathfrak{f}_{*d}\mathfrak{f}_d^{-1}.$$

$\square$

II.5.2 Geometric embeddings

We now establish a bijective correspondence between internal nuclei and Lawvere-Tierney topologies, and hence between internal sublocales and subtopoi. Let $\mathcal{F}$ be a topos. Recall, from [63, §A4.4] say, that a *Lawvere-Tierney topology* is a endomorphism $j\colon \Omega_{\mathcal{F}} \to \Omega_{\mathcal{F}}$ on the subobject classifier of the topos $\mathcal{F}$ such that the diagrams

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega_{\mathcal{F}} \arrow[d, "j"] \\ & \Omega_{\mathcal{F}}, \end{tikzcd} \qquad \begin{tikzcd} \Omega_{\mathcal{F}} \arrow[r, "j"] \arrow[dr, "j"'] & \Omega_{\mathcal{F}} \arrow[d, "j"] \\ & \Omega_{\mathcal{F}}, \end{tikzcd} \qquad \begin{tikzcd} \Omega_{\mathcal{F}} \times \Omega_{\mathcal{F}} \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \Omega_{\mathcal{F}} \arrow[d, "j"] \\ \Omega_{\mathcal{F}} \times \Omega_{\mathcal{F}} \arrow[r, "\wedge"] & \Omega_{\mathcal{F}} \end{tikzcd}$$

commute. Recall also that there is a bijection between Lawvere-Tierney topologies and subtopoi of $\mathcal{F}$. As observed in [79, Corollary IX.6.6], given a locale L, there is a bijection between Lawvere-Tierney topologies on $\Omega_{\mathbf{Sh}(L)}$ (and hence subtopoi of $\mathbf{Sh}(L)$) and nuclei on L (and hence sublocales of L). The following result extends this bijection to the internal setting.

Theorem II.34. *Let $\mathbb{L}\colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ be an internal locale of $\mathcal{E} \simeq \mathbf{Sh}(C, J)$. There is a bijective correspondence between the following*

(i) the subtopoi of $\mathcal{F} \simeq \mathbf{Sh}(\mathbb{L})$;

(ii) the internal nuclei on $\mathbb{L}$;

(iii) the internal sublocales of $\mathbb{L}$.

In particular, if $\mathfrak{f}\colon \mathbb{L}' \to \mathbb{L}$ is an internal locale morphism, $\mathbf{Sh}(\mathfrak{f})$ is a geometric embedding if and only if $\mathfrak{f}$ is an internal locale embedding.

Proof. The bijective correspondence between internal nuclei and internal sublocales was shown in Corollary II.33. We now demonstrate a bijective correspondence between the internal nuclei on $\mathbb{L}$ and the Lawvere-Tierney topologies on $\Omega_{\mathbf{Sh}(L)}$.

Let $j\colon \Omega_{\mathbf{Sh}(\mathbb{L})} \to \Omega_{\mathbf{Sh}(\mathbb{L})}$ be a Lawvere-Tierney topology and let $f\colon \mathbf{Sh}(\mathbb{L}) \to \mathbf{Sets}^{C^{\mathrm{op}}}$ be the localic geometric morphism such that $f_*(\Omega_{\mathbf{Sh}(\mathbb{L})}) \cong \mathbb{L}$, i.e. $f \cong C_{\pi_{\mathbb{L}}}$. By now applying the direct image functor $f_*\colon \mathbf{Sh}(\mathbb{L}) \to \mathbf{Sets}^{C^{\mathrm{op}}}$, we obtain an endomorphism

$$f_*j\colon f_*(\Omega_{\mathbf{Sh}(\mathbb{L})}) \cong \mathbb{L} \longrightarrow f_*(\Omega_{\mathbf{Sh}(\mathbb{L})}) \cong \mathbb{L}.$$

By the description of $C_{\pi_{\mathbb{L}}*}$ afforded by [79, Theorem VII.10.2], we have that

$$(f_*j)_c = (C_{\pi_{\mathbb{L}}*}j)_c = (j \circ t_{\mathbb{L}})_c = j_{(c,\top)}.$$

We claim that f_*j is an internal nucleus.

Since j is a Lawvere-Tierney topology, f_*j makes the following diagrams

$$\begin{tikzcd} \mathbb{L} \arrow[r, "f_*j"] \arrow[dr, "f_*j"'] & \mathbb{L} \arrow[d, "f_*j"] \\ & \mathbb{L}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L} \times \mathbb{L} \arrow[r, "\wedge"] \arrow[d, "f_*j \times f_*j"'] & \mathbb{L} \arrow[d, "f_*j"] \\ \mathbb{L} \times \mathbb{L} \arrow[r, "\wedge"] & \mathbb{L} \end{tikzcd}$$

commute. Thus, $f_*j\colon \mathbb{L} \to \mathbb{L}$ is a natural transformation such that, for each $c \in C$, the component $(f_*j)_c\colon \mathbb{L}_c \to \mathbb{L}_c$ is idempotent and preserves binary meets. It remains to show that $(f_*j)_c$ is inflationary.

Let $U \in \mathbb{L}_c$. As j is a Lawvere-Tierney topology and natural, there is a commutative diagram of sets

$$\begin{tikzcd} \mathbf{1}(c,U) \arrow[r, "\top_{(c,U)}"] \arrow[dr, "\top_{(c,U)}"'] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[d, "j_{(c,U)}"] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,\top) \arrow[l, "-\wedge U"'] \arrow[d, "(f_*j)_c = j_{(c,\top)}"] \\ & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,\top). \arrow[l, "-\wedge U"'] \end{tikzcd}$$

The displayed morphisms act as follows:

(i) the map $\top_{(c,U)}\colon \mathbf{1}(c,U) \to \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U)$ picks out the top element

$$U \in \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \cong \mathrm{Sub}_{\mathbf{Sh}(\mathbb{L})}(\ell_{C\rtimes\mathbb{L}}(c,U)),$$

(ii) while the map $\Omega_{\mathbf{Sh}(\mathbb{L})}(c,\top) \to \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U)$ is induced by pulling back subobjects along the monomorphism $\ell_{C\rtimes\mathbb{L}}(c,U) \rightarrowtail \ell_{C\rtimes\mathbb{L}}(c,\top)$. In other words, it acts by

$$V \mapsto V \wedge U.$$

Thus, by chasing the element $U \in \Omega_{\mathbf{Sh}(L)}(c,\top)$ through the diagram, we deduce that $U \wedge (f_*j)_c(U) = j_U(U) = U$. Thus, $U \leqslant (f_*j)_c(U)$ as desired. Hence, $f_*j\colon \mathbb{L} \to \mathbb{L}$ is a natural transformation in which each component is a nucleus, i.e. f_*j is an internal locale.

Conversely, given an internal nucleus

$$k\colon \mathbb{L} \cong f_*(\Omega_{\mathbf{Sh}(\mathbb{L})}) \longrightarrow \mathbb{L} \cong f_*(\Omega_{\mathbf{Sh}(\mathbb{L})}),$$

we define a natural endomorphism k^f on the subobject classifier $\Omega_{\mathbf{Sh}(\mathbb{L})}$, viewed as a sheaf on the site $(C \rtimes \mathbb{L}, K_{\mathbb{L}})$, by

$$k^f_{(c,U)}(V) = k_c(V) \wedge U,$$

for each $(c,U) \in C \rtimes \mathbb{L}$ and each $V \in \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) = \{\, V \in \mathbb{L}_c \mid V \leqslant U \,\}$. We now demonstrate that k^f defines an Lawvere-Tierney topology.

As k is an internal nucleus, by a simple diagram chase it is clear that, for each object $(c,U) \in C \rtimes \mathbb{L}$, the diagrams

$$\begin{tikzcd} \mathbf{1}(c,U) \arrow[r, "\top_{(c,U)}"] \arrow[dr, "\top_{(c,U)}"'] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[d, "k^f_{(c,U)}"] \\ & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U), \end{tikzcd} \qquad \begin{tikzcd} \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[r, "k^f_{(c,U)}"] \arrow[dr, "k^f_{(c,U)}"'] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[d, "k^f_{(c,U)}"] \\ & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U), \end{tikzcd}$$

$$\begin{tikzcd} \Omega_{\mathbf{Sh}(\mathbb{L})} \times \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[r, "\wedge"] \arrow[d, "k^f_{(c,U)} \times k^f_{(c,U)}"'] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[d, "k^f_{(c,U)}"] \\ \Omega_{\mathbf{Sh}(\mathbb{L})} \times \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \arrow[r, "\wedge"] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c,U) \end{tikzcd}$$

all commute. It remains to observe that k^f is natural. Since each arrow $(c, U) \xrightarrow{g} (d, V)$ of $C \rtimes \mathbb{L}$ can be factored as

$$(c, U) \xrightarrow{\mathrm{id}_c} (c, g^{-1}(V)) \xrightarrow{g} (d, V),$$

it suffices to show that both squares in the diagram

$$\begin{tikzcd}
\Omega_{\mathbf{Sh}(\mathbb{L})}(d, V) \arrow[r] \arrow[d, "k^f_{(d,V)}"] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c, g^{-1}(V)) \arrow[r] \arrow[d, "k^f_{(c,g^{-1}(V))}"] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c, U) \arrow[d, "k^f_{(c,U)}"] \\
\Omega_{\mathbf{Sh}(\mathbb{L})}(d, V) \arrow[r] & \Omega_{\mathbf{Sh}(\mathbb{L})}(d, g^{-1}(V)) \arrow[r] & \Omega_{\mathbf{Sh}(\mathbb{L})}(c, U)
\end{tikzcd}$$

commute.

(i) The left-hand square commutes since, for each $W \in \mathbb{L}_d$,

$$k_c(g^{-1}(W)) \wedge g^{-1}(V) = g^{-1}(k_d(W)) \wedge g^{-1}(V) = g^{-1}(k_d(W) \wedge V).$$

(ii) Meanwhile, the right-hand square commutes since, for each $W \in \mathbb{L}_c$,

$$k_c(W \wedge U) \wedge U = k_c(W) \wedge U = k_c(W \wedge g^{-1}(V)) \wedge U.$$

Finally, the bijection is completed by noting that the two constructions are mutually inverse. That is, for each $c \in C$ and $U, V \in \mathbb{L}_c$,

$$(f_* k^f)_c(V) = k^f_{(c,\top)}(V) = k_c(V) \wedge \top = k_c(V)$$

and

$$(f_* j)^f_{(c,U)}(V) = j_{(c,\top)}(V) \wedge U = j_{(c,U)}(V),$$

for each internal nucleus k on $\mathbb{L}$ and each Lawvere-Tierney topology j on $\Omega_{\mathbf{Sh}(\mathbb{L})}$. □

II.5.3 The surjection-inclusion factorisation

Recall that every locale morphism $f : L \to K$ can be factored uniquely (up to isomorphism) as a surjection of locales followed by an inclusion of locales (see [79, §IX.4]). The same is true for geometric morphisms: every geometric morphism $f : \mathcal{F} \to \mathcal{E}$ can be factored as a geometric surjection composed with an inclusion of a subtopos (see [63, Theorem A4.2.10]). If f is induced by an internal locale morphism, a simple application of Proposition II.28 and Theorem II.34 yields the following.

Corollary II.35. *Let $\mathfrak{f} : \mathbb{L}' \to \mathbb{L}$ be an internal locale morphism of $\mathbf{Sh}(C, J)$. The surjection-inclusion factorisation of the geometric morphism*

$$\mathbf{Sh}(\mathfrak{f}) : \mathbf{Sh}(\mathbb{L}') \to \mathbf{Sh}(\mathbb{L})$$

is induced by the 'pointwise' surjection-inclusion factorisation of $\mathfrak{f}$.

Proof. Let

$$\mathbf{Sh}(\mathbb{L}') \twoheadrightarrow \mathbf{Sh}(\mathbb{L}^{\mathfrak{f}_*\mathfrak{f}^{-1}}) \rightarrowtail \mathbf{Sh}(\mathbb{L})$$

denote the surjection inclusion factorisation of $\mathbf{Sh}(\mathfrak{f})$. By Proposition II.28 and Theorem II.34, the factor $\mathbf{Sh}(\mathbb{L}') \twoheadrightarrow \mathbf{Sh}(\mathbb{L}^{\mathfrak{f}_*\mathfrak{f}^{-1}})$ is induced by a surjective internal locale morphism $\mathbb{L}' \twoheadrightarrow \mathbb{L}^{\mathfrak{f}_*\mathfrak{f}^{-1}}$, while $\mathbf{Sh}(\mathbb{L}^{\mathfrak{f}_*\mathfrak{f}^{-1}}) \rightarrowtail \mathbf{Sh}(\mathbb{L})$ is induced by an internal embedding of locales $\mathbb{L}^{\mathfrak{f}_*\mathfrak{f}^{-1}} \rightarrowtail \mathbb{L}$. Since internal surjections and embeddings are computed 'pointwise', the component at $c \in C$ of these internal locale morphisms must agree with the 'pointwise' surjection-inclusion factorisation of the locale morphism $\mathfrak{f}_c : \mathbb{L}'_c \to \mathbb{L}_c$. □

II.6 The frame of internal nuclei

In this final section, we consider the poset of internal nuclei on an internal locale. It is well-known that this forms a frame, but we show additionally that the frame operations can be computed 'pointwise'.

Let L be a locale and let $N(L)$ denote the set of nuclei on L. We can order $N(L)$ by setting $j \leqslant k$ if $j(U) \leqslant k(U)$ for all $U \in L$. Recall, from [60, Proposition II.2.5] say, that so ordered $N(L)$ is a frame. The set of sublocales of L, written as $\mathrm{Sub}_{\mathbf{Loc}}(L)$, can also be ordered with $[K \rightarrowtail L] \leqslant [K' \rightarrowtail L]$ if and only if there is a factorisation

$$\begin{tikzcd} K \arrow[rr] \arrow[dr, tail] & & K' \arrow[dl, tail] \\ & L. & \end{tikzcd}$$

Under the bijection between nuclei and sublocales, this is precisely the order dual $N(L) \cong \mathrm{Sub}_{\mathbf{Loc}}(L)^{\mathrm{op}}$, and hence $\mathrm{Sub}_{\mathbf{Loc}}(L)$ is a co-frame.

Definitions II.36. Let $\mathbb{L} \colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ be an internal locale of $\mathbf{Sh}(C, J)$, and let $\mathcal{E}$ be a topos.

(i) By $N(\mathbb{L})$ we denote the poset of internal nuclei on $\mathbb{L}$ ordered by $j \leqslant k$ if and only if, for each $c \in C$ and $U \in \mathbb{L}_c$, $j_c(U) \leqslant k_c(U)$.

(ii) By $\mathbf{LT}(\mathcal{E})$ we denote the poset of Lawvere-Tierney topologies for $\mathcal{E}$, ordered by $j \leqslant k$ if and only if $j = j \wedge k$ (this poset is denoted as $\mathbf{Lop}(\mathcal{E})$ in [63, §A4.5]).

(iii) By $\mathrm{Sub}_{\mathbf{Topos}}(\mathcal{E})$ we denote the poset of subtopoi of $\mathcal{E}$ which have been ordered by $[\mathcal{F}' \rightarrowtail \mathcal{E}] \leqslant [\mathcal{F} \rightarrowtail \mathcal{E}]$ if and only if there is a factorisation of geometric morphisms

$$\begin{tikzcd} \mathcal{F}' \arrow[rr] \arrow[dr, tail] & & \mathcal{F} \arrow[dl, tail] \\ & \mathcal{E}. & \end{tikzcd}$$

(iv) By $\mathrm{Sub}_{\mathbf{Loc}(\mathbf{Sh}(C,J))}(\mathbb{L})$ we denote the poset of internal sublocales of $\mathbb{L}$ ordered by $[\mathbb{L}' \rightarrowtail \mathbb{L}] \leqslant [\mathbb{L}'' \rightarrowtail \mathbb{L}]$ if and only if there is a factorisation of internal locale morphisms

$$\begin{tikzcd} \mathbb{L}' \arrow[rr] \arrow[dr, tail] & & \mathbb{L}'' \arrow[dl, tail] \\ & \mathbb{L}. & \end{tikzcd}$$

Under the bijections established in Theorem II.34, there is an isomorphism of posets

$$N(\mathbb{L}) \cong \mathbf{LT}(\mathbf{Sh}(\mathbb{L})) \cong \mathrm{Sub}_{\mathbf{Topos}}(\mathbf{Sh}(\mathbb{L}))^{\mathrm{op}} \cong \mathrm{Sub}_{\mathbf{Loc}(\mathbf{Sh}(C,J))}(\mathbb{L})^{\mathrm{op}}$$

We know already that $\mathrm{Sub}_{\mathbf{Topos}}(\mathbf{Sh}(\mathbb{L}))$ is a complete co-Heyting algebra, i.e. a co-frame (see [63, §A4.5]). We will give an alternative proof using internal nuclei that $N(\mathbb{L})$ is a frame. Our construction demonstrates that the frame operations of $N(\mathbb{L})$ can be computed 'pointwise'. That is, for each subset $\{ j^i \mid i \in I \} \subseteq N(\mathbb{L})$ and each object c of $\mathcal{C}$, there are equalities

$$\left(\bigwedge_{i \in I} j^i \right)_c = \bigwedge_{i \in I} j^i_c, \quad \left(\bigvee_{i \in I} j^i \right)_c = \bigvee_{i \in I} j^i_c,$$

where $\bigwedge_{i \in I} j^i_c$ and $\bigvee_{i \in I} j^i_c$ are the meets and joins as computed in $N(\mathbb{L}_c)$. The first of these equalities is easily shown.

Lemma II.37. *The meet of a subset* $\{ j^i \mid i \in I \} \subseteq N(\mathbb{L})$ *is given by*

$$\left(\bigwedge_{i \in I} j^i \right)_c (U) = \bigwedge_{i \in I} j^i_c(U), \tag{II.v}$$

for each $c \in \mathcal{C}$ *and* $U \in \mathbb{L}_c$.

Proof. If (II.v) defines a valid internal nucleus on $\mathbb{L}$, it must clearly be the meet of the subset $\{ j^i \mid i \in I \} \subseteq N(\mathbb{L})$. Recall from [60, Proposition II.2.5] that $\bigwedge_{i \in I} j^i_c$ yields a nucleus on $\mathbb{L}_c$, for each object $c \in \mathcal{C}$. We must show naturality. As $g^{-1} \colon \mathbb{L}_d \to \mathbb{L}_c$ is open, for an arrow $c \xrightarrow{g} d$ of $\mathcal{C}$, it preserves all meets and so

$$g^{-1} \left(\bigwedge_{i \in I} j^i_d(U) \right) = \bigwedge_{i \in I} g^{-1} j^i_d(U) = \bigwedge_{i \in I} j^i_c g^{-1}(U).$$

Thus, $\bigwedge_{i \in I} j^i$ defines an internal nucleus on $\mathbb{L}$. □

We will demonstrate that $N(\mathbb{L})$ is a frame by generalising the notion of a pre-nucleus on a locale, recalled below, to the internal setting.

Remark II.38. We give some justification as to why the frame operations can be computed 'pointwise' as described in Theorem II.42 below. Recall that the subtopoi of $\mathbf{Sh}(\mathcal{D}, J)$ correspond to Grothendieck topologies J' on $\mathcal{D}$ that contain J. In the case of a Grothendieck topology J' on $\mathcal{C} \rtimes \mathbb{L}$ that contains $K_{\mathbb{L}}$, we observe that the added data is generated by new covering families on the fibres $\mathbb{L}_c$. Specifically, adding a new covering family

$$\left\{ (c_i, U_i) \xrightarrow{f_i} (c, U) \,\middle|\, i \in I \right\}$$

to $K_{\mathbb{L}}$ is equivalent to requiring that the family

$$\left\{ (c, \exists_{f_i} U_i) \xrightarrow{\mathrm{id}_c} (c, U) \,\middle|\, i \in I \right\}$$

is covering.

II.6.1 Pre-nuclei of locales

There are many proofs of the fact that $N(L)$ is a frame for each locale L. For example, the proof found in [60, Proposition II.2.5] shows that $N(L)$ is a complete Heyting algebra by defining the Heyting operation. Alternative approaches using pre-nuclei are considered in [108] and [35]. We will follow the argument of [108] when developing our internal generalisation. We briefly repeat the argument for locales below.

Recall from [108, §2] that a pre-nucleus on a locale L is a (necessarily monotone) map $p\colon L \to L$ that is inflationary and finite-meet-preserving, i.e. for all $U, V \in L$,

$$U \leqslant p(U), \quad p(U \wedge V) = p(U) \wedge p(V).$$

Thus, a nucleus on L is an idempotent pre-nucleus. Unlike nuclei, pre-nuclei are evidently closed under composition.

We denote by $PN(L)$ the poset of pre-nuclei on L ordered by $p \leqslant q$ if $p(U) \leqslant q(U)$ for all $U \in L$. It is clear that $PN(L)$ is a complete lattice. For each subset $\{ p^i \mid i \in I \}$ of $PN(L)$ and each $U \in L$,

$$\left(\bigwedge_{i\in I} p^i\right)(U) = \bigwedge_{i\in I} p^i(U), \quad \left(\bigvee_{i\in I} p^i\right)(U) = \bigvee_{i\in I} p^i(U),$$

where $\bigwedge_{i\in I} p^i(U)$ and $\bigvee_{i\in I} p^i(U)$ are calculated as in L. It follows by the infinite distributive law for L that $PN(L)$ is also a frame.

The inclusion of nuclei into pre-nuclei $N(L) \hookrightarrow PN(L)$ has a left adjoint

$$(-)^\infty\colon PN(L) \to N(L),$$

which we call the *nucleation* (the *nuclear reflection* in [35] and *idempotent closure* in [108]), constructed as follows. For each ordinal α and limit ordinal λ, we define inductively

$$p^0(U) = U, \quad p^{\alpha+1}(U) = p(p^\alpha(U)), \quad p^\lambda(U) = \bigvee_{\alpha<\lambda} p^\alpha(U).$$

At each stage, the resultant map $p^\kappa\colon L \to L$ is a pre-nucleus. Necessarily, as L is small, there is a sufficiently large ordinal κ such that p^κ is idempotent and therefore a nucleus. We label this by p^∞. We observe that if $p \leqslant q$ then $p^\infty \leqslant q^\infty$, that $p \leqslant p^\infty$, and if j is a nucleus then $j = j^\infty$. That is, nucleation is functorial, and has units and counits yielding the adjunction

$$\begin{tikzcd} N(L) \arrow[r, hook] \arrow[r, phantom, bend left=20, "\perp"] & PN(L) \arrow[l, bend right=40, "(-)^\infty"'] \end{tikzcd}$$

witnessing $N(L)$ as a reflective subcategory of $PN(L)$.

Thus, the poset $N(L)$, in addition to the meets constructed in Lemma II.37, has all joins. For a subset

$$\{ j^i \mid i \in I \} \subseteq N(L),$$

the join in $N(L)$ is given by $\left(\bigvee_{i\in I} j^i\right)^\infty$. The infinite distributive law for $N(L)$, and hence the fact that $N(L)$ is a frame, is a consequence of the distributive law for $PN(L)$ and Lemma II.39 below (the lemma is equivalent to [108, Lemma 3.1]).

Lemma II.39. *Let L be a locale, n a nucleus on L, and let $\{p^i \mid i \in I\}$ be a collection of pre-nuclei on L. The infinite distributive law*

$$\left(n \wedge \bigvee_{i\in I} p^i\right)^\infty = n \wedge \left(\bigvee_{i\in I} p^i\right)^\infty$$

holds.

Proof. We will show that

$$\left(n \wedge \bigvee_{i\in I} p^i\right)^{\kappa} = n \wedge \left(\bigvee_{i\in I} p^i\right)^{\kappa},$$

for each ordinal κ, and thereby deduce the result. The base case

$$\left(n \wedge \bigvee_{i\in I} p^i\right)^{0} = \mathrm{id}_L = n \wedge \left(\bigvee_{i\in I} p^i\right)^{0}$$

is trivial, since $U \leqslant n(U)$ for all $U \in L$.

We now perform the inductive step. Suppose that

$$\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha} = n \wedge \left(\bigvee_{i\in I} p^i\right)^{\alpha},$$

then there is a chain of equalities

$$\begin{aligned}
\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha+1} &= \left(n \wedge \bigvee_{i\in I} p^i\right)\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha}, \\
&= n\left(\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha}\right) \wedge \bigvee_{i\in I} p^i\left(\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha}\right), \\
&= nn \wedge n\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right) \wedge \bigvee_{i\in I} p^i n \wedge p^i\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right).
\end{aligned}$$

Using that $nn = n$, $n \leqslant n\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right)$, and $n \leqslant p^i n$, for all i, we have that

$$\begin{aligned}
\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha+1} &= n \wedge \bigvee_{i\in I} p^i n \wedge p^i\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right), \\
&= \bigvee_{i\in I} n \wedge p^i n \wedge p^i\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right), \\
&= \bigvee_{i\in I} n \wedge p^i\left(\left(\bigvee_{i\in I} p^i\right)^{\alpha}\right), \\
&= n \wedge \left(\bigvee_{i\in I} p^i\right)^{\alpha+1}.
\end{aligned}$$

Finally, we perform the limit inductive step. If λ is a limit ordinal such that

$$\left(n \wedge \bigvee_{i\in I} p^i\right)^{\alpha} = n \wedge \left(\bigvee_{i\in I} p^i\right)^{\alpha}$$

for each ordinal $\alpha < \lambda$, then there is a chain of equalities

$$\begin{aligned}\left(n \wedge \bigvee_{i\in I} p^i\right)^\lambda &= \bigvee_{\alpha<\lambda}\left(n \wedge \bigvee_{i\in I} p^i\right)^\alpha ,\\ &= \bigvee_{\alpha<\lambda} n \wedge \left(\bigvee_{i\in I} p^i\right)^\alpha ,\\ &= n \wedge \left(\bigvee_{i\in I} p^i\right)^\lambda .\end{aligned}$$

Hence, by transfinite induction, the result holds. ◻

II.6.2 Internal pre-nuclei

We now extend the theory of pre-nuclei and nucleation to the internal context. In doing so we will observe that $N(\mathbb{L})$ is a frame for every internal locale.

Definition II.40. Let $\mathbb{L}$ be an internal locale of $\mathbf{Sh}(C, J)$. An *internal pre-nucleus* is a natural transformation $p\colon \mathbb{L} \to \mathbb{L}$ such that $p_c\colon \mathbb{L}_c \to \mathbb{L}_c$ is a pre-nucleus, for each $c \in C$. The set of internal pre-nuclei, denoted by $PN(\mathbb{L})$, can be ordered by $p \leqslant q$ if $p_c(U) \leqslant q_c(U)$ for all $c \in C$ and $U \in \mathbb{L}_c$.

It is easily checked that the poset of internal pre-nuclei $PN(\mathbb{L})$ on an internal locale $\mathbb{L}$ of $\mathbf{Sh}(C, J)$ has all meets and all joins, which are computed 'pointwise'. Thus, by the infinite distributivity law for $\mathbb{L}_c$, for each $c \in C$, $PN(\mathbb{L})$ is a frame. We show that an internal nucleation can also be performed 'pointwise'.

Lemma II.41. *Let $p\colon \mathbb{L} \to \mathbb{L}$ be an internal pre-nucleus on an internal locale $\mathbb{L}$, fibred over a category C. The pointwise nucleations $p_c^\infty\colon \mathbb{L} \to \mathbb{L}_c$ of each component p_c of p are the components of an internal nucleus.*

Proof. For each object $c \in C$, the nucleation $p_c^\infty\colon \mathbb{L}_c \to \mathbb{L}_c$ of p_c is a nucleus, so it remains only show that they are natural in c. This is easily shown by transfinite induction. We will perform the case for a limit ordinal λ. Let $c \xrightarrow{g} d$ be an arrow of C. If, for all $\alpha < \lambda$, the square

$$\begin{tikzcd} \mathbb{L}_d \arrow[r, "g^{-1}"] \arrow[d, "p_d^\alpha"'] & \mathbb{L}_c \arrow[d, "p_c^\alpha"] \\ \mathbb{L}_d \arrow[r, "g^{-1}"] & \mathbb{L}_c \end{tikzcd}$$

commutes, then we have the desired equality

$$g^{-1}\left(\bigvee_{\alpha<\lambda} p_d^\alpha\right) = \bigvee_{\alpha<\lambda} g^{-1}p_d^\alpha = \bigvee_{\alpha<\lambda} p_c^\alpha g^{-1}.$$

◻

As a result, we obtain a left adjoint to the inclusion $N(\mathbb{L}) \hookrightarrow PN(\mathbb{L})$,

$$\begin{tikzcd} N(\mathbb{L}) \arrow[r, hook, ""{name=A}] & PN(\mathbb{L}), \arrow[l, bend right, "(-)^{\infty}"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

just as we did for locales. The functor $(-)^{\infty}\colon PN(\mathbb{L}) \to N(\mathbb{L})$, the *internal nucleation*, sends internal pre-nuclei to their pointwise nucleation.

Theorem II.42. *Let $\mathbb{L}$ be an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$. The poset $N(\mathbb{L})$ of internal nuclei is a frame whose frame operations can be computed 'pointwise', by which we mean that, for each subset $\{ j^i \mid i \in I \} \subseteq N(\mathbb{L})$ and each object c of $\mathcal{C}$, there are equalities*

$$\left(\bigwedge_{i\in I} j^i\right)_c = \bigwedge_{i\in I} j^i_c, \text{ and } \left(\bigvee_{i\in I} j^i\right)_c = \bigvee_{i\in I} j^i_c, \tag{II.vi}$$

where $\bigwedge_{i\in I} j^i_c$ and $\bigvee_{i\in I} j^i_c$ are computed as in $N(\mathbb{L}_c)$.

Proof. We saw in Lemma II.32 that $N(\mathbb{L})$ has all meets and that these are computed pointwise. The join of $\{ j^i \mid i \in I \} \subseteq N(\mathbb{L})$ is the nucleation of the join of $\{ j^i \mid i \in I \}$ as internal pre-nuclei. Since the nucleation of internal pre-nuclei is computed pointwise, as are joins in $PN(\mathbb{L})$, the joins in $N(\mathbb{L})$ are also computed pointwise in the sense of (II.vi). Finally, as $N(\mathbb{L}_c)$ satisfies the infinite distributivity law for each $c \in \mathcal{C}$, we obtain the infinite distributivity law for $N(\mathbb{L})$. □

Since every topos $\mathcal{E}$ is the topos of sheaves $\mathbf{Sh}(\mathbb{L})$ for some internal locale $\mathbb{L}$ (see, for example, [68, Proposition VII.3.1]), and also because $\mathrm{Sub}_{\mathbf{Topos}}(\mathcal{E}) \cong N(\mathbb{L})^{\mathrm{op}}$, we have recovered the well-known fact that the poset of subtopoi of a topos is a co-frame.

Remark II.43. Let $\mathbb{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ be an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$. Since the frame operations of $N(\mathbb{L})$ are computed 'pointwise', for each object c of $\mathcal{C}$, the projection $\pi_c\colon N(\mathbb{L}) \to N(\mathbb{L}_c)$ that sends an internal nucleus $j\colon \mathbb{L} \to \mathbb{L}$ to its component j_c at c preserves all joins and meets. Therefore,

$$\pi_c\colon N(\mathbb{L}) \longrightarrow N(\mathbb{L}_c)$$

is an open frame homomorphism.

Chapter III

Classifying topoi via doctrines

What is doctrine theory? Doctrines (also called indexed or fibred preorders), as introduced by Lawvere in [77] and expanded upon in [78], are a natural categorical setting in which to study first-order logic, as evidenced by the prototypical examples derived from theories recalled in Section III.1 below. Their relation to logic can be summarised as: doctrines are to first-order theories as Lindenbaum-Tarski algebras are to propositional theories.

Doctrines are a powerful tool within categorical logic as the doctrine of a theory can be seen to express certain logical syntax, interpreted by categorical constructions, even when this is not present in the explicit symbolic syntax of the logic. An example is given in Example III.12.

What is the classifying topos of a doctrine? In addition to the syntax of a theory, there is an intuitive notion of the semantics of the theory as well – the mathematical objects, and their morphisms, described by the theory. The notion of model for a theory extends naturally to a notion of model for a doctrine P (once some choice about the pertinent structure of P is made).

Models do not need to be set-based. The notion of model for a doctrine P can be extended to any arbitrary topos $\mathcal{E}$, yielding a category $P\text{-mod}(\mathcal{E})$ of models of P internal to $\mathcal{E}$. Just as for theories, a *classifying topos* for a doctrine P is defined as a topos $\mathcal{E}_P$ for which there is a natural equivalence $\mathbf{Geom}(\mathcal{E}, \mathcal{E}_P) \simeq P\text{-mod}(\mathcal{E})$ for each topos $\mathcal{E}$. Evidently, whenever a classifying topos exists for P, it is unique up to equivalence.

Our goal. This chapter exposits a doctrinal approach to classifying topos theory. We aim to show that, if a theory $\mathbb{T}$ has a classifying topos $\mathcal{E}$, then the process $\mathbb{T} \mapsto \mathcal{E}_{\mathbb{T}}$ can be factored by first sending the theory to its associated doctrine, and then sending said doctrine to its classifying topos, as displayed in the schematic

$$\begin{tikzcd}
\textbf{Theories} \arrow[rr, "\substack{\text{Doctrine associated}\\ \text{with a theory}}"] \arrow[dr, bend right, "\substack{\text{Classifying topos}\\ \text{of a theory}}"'] & & \textbf{Doctrines} \arrow[dl, bend left, "\substack{\text{Classifying topos}\\ \text{of a doctrine.}}"] \\
& \textbf{Topoi} &
\end{tikzcd} \tag{III.i}$$

Note that these are 'processes' or 'constructions' and not functors. Indeed, we will not seek to make sense of a 'morphism of theories' here (although the latter process **Doctrines** → **Topoi** can be made suitably functorial).

However, in Example III.12 we give an example of a doctrine obtained two inequivalent theories over two different syntaxes. Thus, the process of sending a theory to its associated doctrine forgets logical structure. We must keep track of this syntax if we are to hope to build the classifying topos of a theory from its associated doctrine.

Our remedy is to use Grothendieck topologies to encode this further logical syntax. We therefore elect to work with *doctrinal sites* rather than doctrines, and our schematic (III.i) becomes

$$\begin{tikzcd}
\begin{array}{c}\textbf{Theories}\\ \textbf{+ syntax}\end{array} \arrow[rr, "\substack{\text{Doctrinal site}\\ \text{associated with a theory}}"] \arrow[dr, bend right, "\substack{\text{Classifying topos}\\ \text{of a theory}}"'] & & \begin{array}{c}\textbf{Doctrinal}\\ \textbf{sites}\end{array} \arrow[dl, bend left, "\substack{\text{Sheaves on a}\\ \text{doctrinal site.}}"] \\
& \textbf{Topoi} &
\end{tikzcd}$$

Focusing exclusively on geometric theories, the above processes restrict to

$$\begin{tikzcd}
\begin{array}{c}\textbf{Geometric}\\ \textbf{theories}\end{array} \arrow[rr, "\substack{\text{Doctrinal site}\\ \text{associated with a theory}}"] \arrow[dr, bend right, "\substack{\text{Classifying topos}\\ \text{of a theory}}"'] & & \begin{array}{c}\textbf{Internal}\\ \textbf{locales}\end{array} \arrow[dl, bend left, "\substack{\text{Sheaves on an}\\ \text{internal locale,}}"] \\
& \textbf{Topoi} &
\end{tikzcd}$$

thereby witnessing a connection between geometric theories and the theory of internal locales studied in Chapter II. Our study of properties of internal locale morphisms in Chapter II will yield elegant alternative proofs of previously known facts in topos theory.

Philosophical motivation. Classifying topos theory can be tersely summarised as: every geometric theory has a classifying topos, and every topos is the classifying topos of some geometric theory. However, this summary can somewhat obfuscate the fact that many non-geometric theories also have classifying topoi (though the latter statement prefigures the *geometric completion* we will study in Chapter IV).

Our purpose in pursuing a doctrinal approach to classifying topos theory is mainly philosophical: we aim to demonstrate how, in an intuitive manner, a classifying topos can be associated with almost any system of predicate reasoning, without prejudice as to the underlying syntax. Moreover, although we won't investigate it here, by Theorem I.21 the classifying topos of a doctrine satisfies a stronger 'relative' universal property than the property we will prove.

Overview. The chapter proceeds as follows.

(A) Varied definitions of doctrine are used in the literature depending on the fragment of logic being interpreted. We fix our terminology in Section III.1. We recall the 'theory to doctrine' construction

$$\textbf{Theories} \longrightarrow \textbf{Doctrines}$$

and recall how, as observed by Lawvere [77], categorical structure on the doctrine corresponds to logical syntax, a relationship made precise in the theorems of Seely [106].

(B) Section III.2 first recalls from [73] the definition of a model of a doctrine. We demonstrate the existence of classifying topoi for a wide class of doctrines. During our discussion on the use of Grothendieck topologies to encode logical syntax, the 2-category of *doctrinal sites* is also introduced.

(C) In Section III.3, we compare the doctrinal approach to classifying topos theory from Section III.2 to the standard construction using syntactic sites found in [87].

(D) Finally, Section III.4 focuses on the relationship between geometric theories, geometric doctrines and internal locales. We show that our study of properties of internal locale morphisms from Chapter II yield elegant proofs of known results on geometric theories.

III.1 Doctrine theory

Doctrines have appeared in many guises throughout the literature, with various assumptions on their structure.

Definition III.1. For us, a doctrine is simply a pseudo-functor

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textbf{PreOrd}.$$

Being fibred categories, doctrines naturally form a 2-category **Doc** as follows.

(i) The objects of **Doc** are doctrines $P\colon \mathcal{C}^{\mathrm{op}} \to \textbf{PreOrd}$.

(ii) An arrow of **Doc** from $P\colon \mathcal{C}^{\mathrm{op}} \to \textbf{PreOrd}$ to $Q\colon \mathcal{D}^{\mathrm{op}} \to \textbf{PreOrd}$ consists of a pair (F, a), where F is a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a is a pseudo-natural transformation $a\colon P \Rightarrow Q \circ F^{\mathrm{op}}$, as in the diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dr, "P"] \arrow[dd, "F^{\mathrm{op}}"'] & \\ \arrow[r, phantom, "\Downarrow a"] & \textbf{PreOrd}. \\ \mathcal{D}^{\mathrm{op}} \arrow[ur, "Q"'] & \end{tikzcd}$$

(iii) A 2-cell $\alpha\colon (F, a) \Rightarrow (F', a')$ is a natural transformation $\alpha\colon F \Rightarrow F'$ such that $a_c(x) \leqslant Q(\alpha_c)(a'_c(x))$ for each object $c \in \mathcal{C}$ and $x \in P(c)$, i.e.

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, bend right, "F^{\mathrm{op}}"'] \arrow[dd, bend left, "F'^{\mathrm{op}}"] \arrow[drr, "P"] & & \\ \arrow[r, phantom, "\xRightarrow{\alpha}"] & a \Downarrow \leqslant \Downarrow a' & \textbf{PreOrd}. \\ \mathcal{D}^{\mathrm{op}} \arrow[urr, "Q"'] & & \end{tikzcd}$$

Remark III.2. Each morphism of doctrines $(F, \mathfrak{a})\colon P \to Q$ induces a functor

$$F \rtimes \mathfrak{a}\colon \mathcal{C} \rtimes P \longrightarrow \mathcal{D} \rtimes Q.$$

We could have chosen the natural transformations $F \rtimes \mathfrak{a} \Rightarrow F' \rtimes \mathfrak{a}'$ as the 2-cells of **Doc**. In the non-pathological cases, the two coincide.

Firstly, every 2-cell $\alpha\colon (F, \mathfrak{a}) \Rightarrow (F', \mathfrak{a}')$ of **Doc** induces a natural transformation $\breve{\alpha}\colon F \rtimes \mathfrak{a} \Rightarrow F' \rtimes \mathfrak{a}'$, with components $\breve{\alpha}_{(c,x)}\colon (F(c), \mathfrak{a}_c(x)) \to (F'(c), \mathfrak{a}'_c(x))$ named by the arrows $\alpha_c\colon F(c) \to F'(c)$.

Conversely, if $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ has non-empty fibres ($P(c) \neq \emptyset$ for all $c \in \mathcal{C}$), then any natural transformation $\beta\colon F \rtimes \mathfrak{a} \Rightarrow F' \rtimes \mathfrak{a}'$ yields a natural transformation $\beta'\colon F \Rightarrow F'$ for which $\mathfrak{a}_c(x) \leqslant Q(\beta'_c)(\mathfrak{a}'_c(x))$ by taking $\beta'_c\colon F(c) \to F'(c)$ as the $\mathcal{D}$-arrow $\beta_{(c,x)}\colon (F(c), \mathfrak{a}_c(x)) \to (F'(c), \mathfrak{a}'_c(x))$, for some $x \in P(c)$.

Example III.3. We can always obtain a doctrine from a cartesian category $\mathcal{C}$ by taking its *doctrine of subobjects*

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat} \subseteq \mathbf{PreOrd},$$

the doctrine where:

(i) for each object c of $\mathcal{C}$, $\mathrm{Sub}_{\mathcal{C}}(c)$ is the meet-semilattice of subobjects of c,

(ii) for each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, $\mathrm{Sub}_{\mathcal{C}}(f)\colon \mathrm{Sub}_{\mathcal{C}}(c) \to \mathrm{Sub}_{\mathcal{C}}(d)$ is the map which sends a subobject $e \rightarrowtail c$ to the pullback

$$\begin{tikzcd} f^*(e) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, tail] \\ d \arrow[r, "f"] & c. \end{tikzcd}$$

In particular, taking the subobject doctrine for the category **Sets** yields the *powerset doctrine* $\mathscr{P}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{CBool} \subseteq \mathbf{PreOrd}$.

The formal definition of a doctrine is motivated by the 'theory-to-doctrine' construction

$$\mathbf{Theories} \longrightarrow \mathbf{Doctrines}$$

of the schematic (III.i) presented below, which assigns a doctrine to every first-order theory. The central conceit behind the *doctrine of a theory* is to collect formulae according to their *context* and then perform a *fibred* Lindenbaum-Tarski construction. Thus, doctrines are the natural algebraic framework with which to study first-order theories in the same way that Lindenbaum-Tarski algebras relate propositional theories with preorders/posets.

III.1.1 From theories to doctrines

Let $\mathbb{T}$ be a theory in a fragment of infinitary first-order logic over a signature Σ with N sorts. We describe how to assign a doctrine $F^{\mathbb{T}}$ to the theory $\mathbb{T}$. We first describe the base category of 'contexts' for the sorts of Σ. Then, we will describe how to add the doctrine structure on top. Finally, following this, we give some examples of how the presence of logical syntactic features can be detected by algebraic properties of the resulting doctrine.

The category of contexts.

Definition III.4. Let Σ be a signature with N sorts. We denote by $\mathbf{Con}_N$ the *category of contexts* for the sorts, the category

(i) whose objects are finite tuples of free variables $\vec{x}$ or *contexts*, i.e. a finite set where each element $x_i \in \vec{x}$ is assigned a sort in Σ,

(ii) and whose arrows are *relabellings* $\sigma \colon \vec{y} \to \vec{x}$, i.e. any function of finite sets such that y_i and $\sigma(y_i)$ have the same sort for all $y_i \in \vec{y}$.

If Σ is a single sorted signature, then $\mathbf{Con}_N \simeq \mathbf{FinSets}$. If Σ has a finite number N of sorts, then $\mathbf{Con}_N \simeq N \times \mathbf{FinSets}$. More generally, $\mathbf{Con}_N$ is the full subcategory of $N \times \mathbf{FinSets}$ on objects $(Z_k)_{k \in N}$ where only finitely many Z_k are non-empty.

Immediately we deduce that $\mathbf{Con}_N$ has all finite limits and colimits, since the category $N \times \mathbf{FinSets}$ has all finite limits and colimits (these being computed pointwise) and the full subcategory $\mathbf{Con}_N \subseteq N \times \mathbf{FinSets}$ is closed under these. A simple generalisation of [79, §VIII.4] yields the following observations about $\mathbf{Con}_N$.

Proposition III.5 (§VIII.4 [79])**.** *Let Σ be a signature with N sorts.*

(i) The category $\mathbf{Con}_N$ is the free category with finite colimits and N generators.

(ii) The presheaf topos $\mathbf{Sets}^{\mathbf{Con}_N}$ classifies the theory $N \cdot \mathbb{O}$ of N objects, i.e. the empty theory over the signature with N sorts.

The doctrine of a theory.

Definition III.6. We denote by $F^{\mathbb{T}} \colon \mathbf{Con}_N \to \mathbf{PoSet}$ the functor that acts as follows.

(i) For each context $\vec{x}$, $F^{\mathbb{T}}(\vec{x})$ is the poset

  a) whose elements are $\mathbb{T}$-provable equivalence classes of formulae in the context $\vec{x}$ (we will abuse notation and not differentiate between a formula and its equivalence class),

  b) and the order relation is given by provability in $\mathbb{T}$, i.e. $\varphi \leqslant \psi$ if and only if $\mathbb{T}$ proves $\varphi \vdash_{\vec{x}} \psi$.

(ii) For each relabelling of contexts $\sigma \colon \vec{y} \to \vec{x}$,

$$F^{\mathbb{T}}(\sigma) \colon F^{\mathbb{T}}(\vec{y}) \longrightarrow F^{\mathbb{T}}(\vec{x})$$

denotes the monotone map that acts by sending a formula $\psi \in F^{\mathbb{T}}(\vec{y})$ to the formula $\psi[\vec{x}/_{\sigma}\vec{y}]$, the formula obtained by simultaneously replacing each instance of the variable $y_i \in \vec{y}$ by $\sigma(y_i) \in \vec{x}$ (since contexts are assumed to be disjoint, we can simultaneously replace free variables).

Notation III.7. In Definition III.6, we have left implicit from which fragment of logic we are taking the formulae which make up the elements $F^{\mathbb{T}}(\vec{x})$. For the most part, we will infer the fragment from the theory. When there could be some confusion, we will denote the fragment in subscript. For example, given a coherent theory $\mathbb{T}$, we denote by

$$F^{\mathbb{T}}_{\mathrm{Coh}} \colon \mathbf{Con}_N \longrightarrow \mathbf{DLat} \subseteq \mathbf{PreOrd}$$

the doctrine whose elements of each fibre are the $\mathbb{T}$-provable equivalence classes of *coherent* formulae, and use

$$F^{\mathbb{T}}_{\mathrm{Geom}} \colon \mathbf{Con}_N \longrightarrow \mathbf{Frm} \subseteq \mathbf{PreOrd}$$

to denote the doctrine where we have used *geometric* formulae instead.

Remark III.8. Let $\mathbb{T}$ be a theory over a signature Σ with N sorts. We have chosen to fibre the doctrine of a theory $\mathbb{T}$ over the category of contexts and relabellings $\mathbf{Con}_N$. Many other choices are also used in the literature. For example, in [30], the following category $\mathbf{Term}_\Sigma$ is used as a base category instead:

(i) the objects of $\mathbf{Term}_\Sigma$ are pairs $\left\langle \vec{x}, \vec{s} = \vec{t} \right\rangle$, where $\vec{s}$ and $\vec{t}$ are terms in Σ (of the same type) whose free variables occur in the context $\vec{x}$;

(ii) an arrow $\left\langle \vec{x}, \vec{s} = \vec{t} \right\rangle \to \langle \vec{y}, \vec{u} = \vec{v} \rangle$ of $\mathbf{Term}_\Sigma$ is a tuple of terms $\vec{w}$, whose free variables are contained in $\vec{x}$, of the same sort as $\vec{y}$, for which $\mathbb{T}$ proves the sequent
$$\vec{s} = \vec{t} \vdash_{\vec{x}} \vec{u}\,[\vec{w}/\vec{y}] = \vec{v}\,[\vec{w}/\vec{y}].$$

Ultimately, these other choices of base category yield *Morita equivalent* doctrines, in the sense that they have equivalent classifying topoi, defined below.

As aforementioned, the simplicity of doctrine theory allows us to encapsulate within our framework not only those theories from familiar fragments of first-order logic, but any system of predicate reasoning. We intuit that a "first-order formal system" $\mathfrak{F}$ ought to be simply a set of rules regarding the manipulation of some (potentially infinite) strings of symbols, which we suggestively call *well-formed formulae*. The key aspect that identifies a formal system as first-order is that each well-formed formula is assigned a *context*, and for each transformation of contexts, there is a *substitution map* that sends well-formed formulae in the domain context to well-formed formulae in the codomain context (normally by replacing certain sub-strings of symbols by other symbols).

It is then clear how to assign a doctrine to the formal system $\mathfrak{F}$, in a manner analogous to the above (though we may wish to refrain from taking equivalence classes of well-formed formulae – see Remark III.9), under the further assumptions that:

(i) the contexts of $\mathfrak{F}$ and their transformations constitute a category;

(ii) in $\mathfrak{F}$, the order $\vdash_c$ on the well-formed formulae in context c, where $\varphi \vdash_c \psi$ if and only if the string ψ is derivable in $\mathfrak{F}$ from the string φ, is reflexive and transitive.

(iii) for this order, each substitution map is monotone.

Remark III.9. In the literature, a doctrine $P \colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ is often assumed to factor through the subcategory $\mathbf{PoSet} \subseteq \mathbf{PreOrd}$. In this case, P is a genuine functor, not only a pseudo-functor.

Of course, $\mathbf{PoSet}$ is a reflective subcategory of $\mathbf{PreOrd}$,

$$\begin{tikzcd} \mathbf{PoSet} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathbf{PreOrd}, \arrow[l, shift right] \end{tikzcd}$$

and so by post-composing a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ with the posetal reflection $\mathbf{PreOrd} \to \mathbf{PoSet}$ yields a universal way of turning P into a **PoSet**-valued doctrine.

In Definition III.6, the doctrine associated to a first-order theory thus obtained is a **PoSet**-valued doctrine because the elements of each fibre are taken as the $\mathbb{T}$-provable equivalence classes of formulae. If simply the formulae are taken instead, the resultant doctrine would be **PreOrd**-valued, not **PoSet**-valued.

We have deliberately left the possibility for **PreOrd**-valued doctrines open for those readers who wish the fibres of their doctrine to be endowed with operations that do not respect provable equivalence. For example, some modal logics possess operators $\Box$ where the equivalence of two formulae $\varphi \equiv \psi$ does not imply that $\Box\varphi \equiv \Box\psi$. These are examples of so-called *non-algebraisable* logics (see [12, §5.2]).

III.1.2 Categorically interpreting logical syntax

As currently formulated, an arbitrary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ does not interpret any particular logical syntax. To model a certain logical syntax, we can require a doctrine to possess the appropriate categorical structure. This can appear as structure on the fibres, as is the case for the logical connectives $\{\bot, \wedge, \vee, \top\}$, or as properties of the substitution maps, in the case of the symbols $\{\exists, =, \forall\}$.

Examples III.10. (i) For any theory $\mathbb{T}$ in a fragment of logic in which conjunctions of formulae and truth are expressible, each fibre of $F^{\mathbb{T}}$ has finite meets. We thus define a *primary doctrine*, a doctrine capable of interpreting the symbols $\{\wedge, \top\}$, as a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ that factors through the subcategory $\mathbf{MSLat} \subseteq \mathbf{PreOrd}$ of meet-semilattices and meet-semilattice homomorphisms. Following the standard terminology found in the literature (say, for instance, [81]), we also assume that $\mathcal{C}$ is a cartesian category (this also ensures that $\mathcal{C} \rtimes P$ is a cartesian category too).

A morphism $(F, a)\colon P \to Q$ in **Doc**, between primary doctrines, is said to be a *morphism of primary doctrines* if the finite limit data are preserved, that is to say: $F\colon \mathcal{C} \to \mathcal{D}$ is left exact and $a_c\colon P(c) \to Q(F(c))$ is a meet-semilattice homomorphism for each $c \in \mathcal{C}$. We denote by **PrimDoc** the 2-full 2-subcategory of **Doc** of primary doctrines and primary doctrine morphisms (by 2-full we mean that the 2-subcategory is full on 2-cells).

(ii) As detailed in [77] and [78], if existential quantification or equality are also expressible , then certain transition maps of the doctrine $F^{\mathbb{T}}$ have left adjoints.

a) Let $\iota\colon \vec{x} \hookrightarrow \vec{x}, \vec{y}$ denote the inclusion of a sub-context (i.e. a coproduct inclusion $\vec{x} \to \vec{x} + \vec{y}$ in $\mathbf{Con}_N$). The rules of existential quantification ensure that the map

$$\begin{aligned} \exists_{F^{\mathbb{T}}(\iota)}\colon F^{\mathbb{T}}(\vec{x}, \vec{y}) &\to F^{\mathbb{T}}(\vec{x}), \\ \varphi &\mapsto \exists \vec{y}\, \varphi, \end{aligned}$$

defines a left adjoint to the substitution map $F^{\mathbb{T}}(\iota)$.

b) Let $\delta\colon \vec{x}, \vec{y}_1, \vec{y}_2 \to \vec{x}, \vec{y}$ denote the relabelling that identifies two identical copies of the tuple $\vec{y}$ (i.e. δ is a co-diagonal $\vec{x} + \vec{y} + \vec{y} \twoheadrightarrow \vec{x} + \vec{y}$ in $\mathbf{Con}_N$). The map

$$\begin{aligned}\exists_{F^{\mathbb{T}}(\delta)} \colon F^{\mathbb{T}}(\vec{x}, \vec{y}_1, \vec{y}_2) &\to F^{\mathbb{T}}(\vec{x}, \vec{y}), \\ \varphi &\mapsto \varphi \wedge \vec{y}_1 = \vec{y}_2\end{aligned}$$

defines a left adjoint to $F^{\mathbb{T}}(\delta)$.

Note that, since every relabelling of contexts can be expressed as a composite of a coproduct inclusion and a co-diagonal, every transition map of $F^{\mathbb{T}}$ has a left adjoint.

An *existential doctrine*, a doctrine capable of interpreting the the logical symbols $\{\wedge, \top, \exists, =\}$, is a primary doctrine where, in addition, $P(f)$ has a left adjoint $\exists_{P(f)}$, for each arrow $d \xrightarrow{f} c \in \mathcal{C}$. We also require that these left adjoints satisfy both the Frobenius and Beck-Chevalley conditions, which express how existential quantification/equality interacts with, respectively, conjunction and substitution under relabellings.

A *morphism of existential doctrines* $(F, a) \colon P \to Q$ is a morphism of primary doctrines which also preserves the interpretation of the new symbols $\{\exists, =\}$, i.e. a morphism for which the square

$$\begin{tikzcd} P(c) \arrow[d, "a_c"'] & P(d) \arrow[l, "\exists_{P(f)}"'] \arrow[d, "a_d"] \\ Q(F(c)) & Q(F(d)) \arrow[l, "\exists_{Q(f)}"'] \end{tikzcd}$$

commutes for each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$. We denote the resultant 2-full 2-subcategory of **Doc** by **ExDoc**.

(iii) A *coherent doctrine* is an existential doctrine that takes values in the category **DLat** of distributive lattices and lattice homomorphisms, and thus interprets the symbols $\{\wedge, \top, \exists, =, \vee, \bot\}$. Morphisms of coherent doctrines are those morphisms (F, a) of existential doctrines where a is a natural transformation of **DLat**-valued functors. We denote the resultant 2-full 2-subcategory of **Doc** by **CohDoc**.

(iv) *Heyting (intuitionistic) doctrines* are coherent doctrines that take values in the category **Heyt** of Heyting algebras and Heyting algebra morphisms, and so interpret (intuitionistically) the symbols $\{\wedge, \top, \exists, =, \vee, \bot, \to\}$. Analogously with coherent doctrines, we can define a subcategory **HeytDoc** of **Doc** of Heyting doctrines and their morphisms.

(v) A *Boolean (classical) doctrine* is a coherent doctrine that takes values in the category **Bool** of Boolean algebras and Boolean algebra homomorphisms, and for each arrow $d \xrightarrow{f} c \in \mathcal{C}$, $P(f)$ also has a right adjoint $\forall_{P(f)}$ expressing universal quantification. Hence, a Boolean doctrine interprets classical first-order logic. A morphism of Boolean doctrines is a morphism (F, a) of coherent doctrines that preserves the interpretation of classical first order logic. We denote the resultant 2-full 2-subcategory of **Doc** by **BoolDoc**.

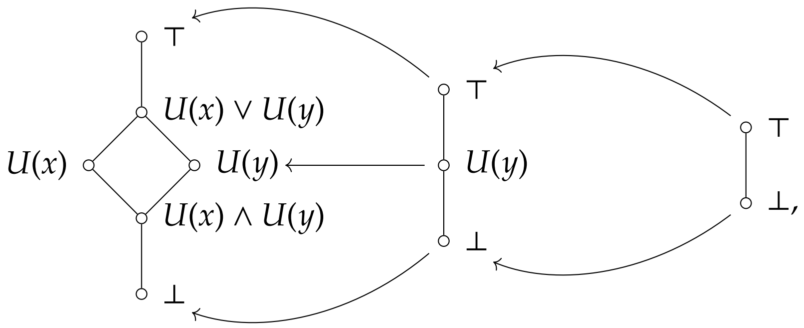
Remark III.11. We have required existential and coherent doctrines to be examples of primary doctrines, but we shall observe in Example III.31 how one might define existential doctrines, etc., over non-cartesian base categories.

Example III.12. By phrasing a study of logical theories algebraically, we are able to recognise the existence of logical structure even when this is not present in the explicit syntax of the theory. We now give a simple example.

Let $\mathbb{E}_U$ be the theory with no axioms over the single-sorted signature with a single unary relation symbol U. The doctrine

$$F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}} \colon \mathbf{FinSets} \longrightarrow \mathbf{PreOrd}$$

associated with the theory and the fragment $\{\perp, \wedge, \vee, \top\}$ can be visualised as

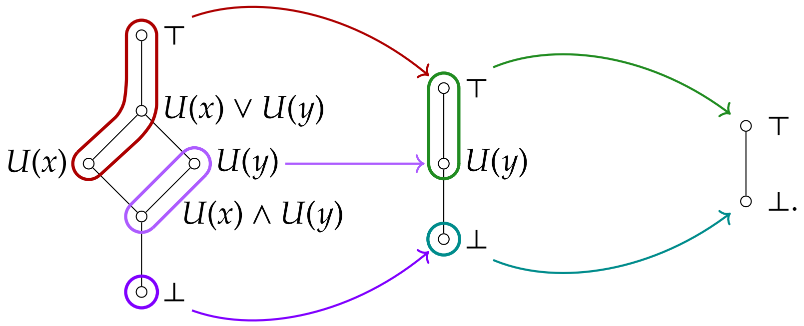


where we have truncated the doctrine $F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}}$ to the subcategory

$$\mathbf{0} \hookrightarrow \mathbf{1} \hookrightarrow \mathbf{2} \subseteq \mathbf{FinSets}.$$

The displayed arrows represent the action on each fibre by the substitution maps $F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}}(\mathbf{0} \hookrightarrow \mathbf{1})$ and $F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}}(\mathbf{1} \hookrightarrow \mathbf{2})$.

We are now able to observe that both of these substitution maps have left adjoints. The action of these left adjoints is represented in the diagram



The elements of each subset are sent by the left adjoints to the target of the corresponding arrow. One can calculate that this defines the action of a pair of left adjoints to the substitution maps.

In fact, for every injective relabelling ι, the map $F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}}(\iota)$ has a left adjoint. Moreover, these left adjoints satisfy Frobenius reciprocity and the Beck-Chevalley condition (wherever appropriate). Therefore, the doctrine $F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}}$ categorically interprets existential quantification, even though our theory is not in a fragment of logic involving quantifiers.

Indeed, there is an isomorphism of doctrines

$$F^{\mathbb{E}_U}_{\{\perp,\wedge,\vee,\top\}} \cong F^{\mathbb{T}}_{\{\perp,\wedge,\vee,\top,\exists\}}$$

where $\mathbb{T}$ is the theory over the same signature as $\mathbb{E}_U$, in the fragment of first-order logic now also including existential quantification, with the single axiom $\top \vdash_{\emptyset} \exists x\, U(x)$.

III.1.3 From doctrines to theories

What does it mean to say a doctrine 'interprets' a certain logical theory? The precise relationship between doctrines and logical theories was elucidated in [106, Theorem 6.1 & Theorem 6.2]. In one direction, given a theory $\mathbb{T}$ over a signature Σ in a certain fragment of logic, the doctrine $F^{\mathbb{T}}$ as defined in Definition III.6 is of the appropriate form. For example, if $\mathbb{T}$ is a coherent theory, i.e. a theory in the fragment of intuitionistic first order logic containing the symbols $\{\wedge, \top, \exists, =, \vee, \bot\}$, then $F^{\mathbb{T}}$ is a coherent doctrine.

The converse relationship is provided by associating a theory to each doctrine fibred over the base category $\mathbf{Con}_N$ as elaborated below. The outputted theory can be chosen to be of the appropriate fragment of first-order logic. For example, if $P\colon \mathbf{Con}_N \to \mathbf{Bool}$ is a Boolean doctrine, then the associated theory $\mathbb{T}_P$ can be chosen to be a theory in the full fragment of (finitary) classical first-order logic.

Definition III.13. Let Σ be a signature with N sorts and let $P\colon \mathbf{Con}_N \to \mathbf{PoSet}$ be a doctrine. We will also assume that P is of an appropriate form, e.g. primary/existential/etc. The *theory associated to* P is the theory $\mathbb{T}_P$ over a signature Σ_P defined in the following way.

(i) The sorts of Σ_P are the same as the sorts of Σ and, for each $U \in P(\vec{x})$, there is a relation symbol R_U with the same type as $\vec{x}$.

(ii) The theory $\mathbb{T}_P$ has as axioms all those sequents expressible in the appropriate fragment of logic (e.g., if P is an existential doctrine, those sequents expressible in *regular logic*, i.e. using the symbols $\{\wedge, \top, \exists, =\}$) which are satisfied by the doctrine P. For example, $\mathbb{T}_P$ contains as an axiom the sequent $R_U \vdash_{\vec{x}} R_V[\vec{x}/_\sigma \vec{y}]$ for each relabelling $\vec{y} \xrightarrow{\sigma} \vec{x} \in \mathbf{Con}_N$ and pair $U \in P(\vec{x})$, $V \in P(\vec{y})$ such that $U \leqslant P(\sigma)(V)$.

These two constructions, i.e. sending a theory $\mathbb{T}$ over Σ to its associated doctrine $F^{\mathbb{T}}\colon \mathbf{Con}_N \to \mathbf{PoSet}$ and a doctrine $P\colon \mathbf{Con}_N \to \mathbf{PoSet}$ to its theory $\mathbb{T}_P$ over Σ_P, are mutually inverse in the sense of the following theorem.

Theorem III.14 (Theorem 6.1 & Theorem 6.2 [106])**.** *Let $\mathbb{T}$ be a theory over a signature Σ with N sorts and let $P\colon \mathbf{Con}_N \to \mathbf{PoSet}$ be a doctrine that interprets the underlying syntax of $\mathbb{T}$.*

(i) There is a natural isomorphism $P \cong F^{\mathbb{T}_P}$.

(ii) The theories $\mathbb{T}$ and $\mathbb{T}_{F^{\mathbb{T}}}$ are equivalent in the sense that:

a) for every formula φ over Σ in the context $\vec{x}$, there exists a canonical choice of formula $\overline{\varphi}$ over $\Sigma_{F^{\mathbb{T}}}$ in the context $\vec{x}$ where, if $\mathbb{T}$ proves the sequent $\varphi \vdash_{\vec{x}} \psi$, then $\mathbb{T}_{F^{\mathbb{T}}}$ proves the sequent $\overline{\varphi} \vdash_{\vec{x}} \overline{\psi}$;

b) for each formula χ over $\Sigma_{F^{\mathbb{T}}}$ in context $\vec{x}$, there exists a formula χ' over Σ in context $\vec{x}$ such that χ and $\overline{\chi'}$ are $\mathbb{T}_{F^{\mathbb{T}}}$-provably equivalent and moreover, if $\mathbb{T}_{F^{\mathbb{T}}}$ proves the sequent $\chi \vdash_{\vec{x}} \xi$, then $\mathbb{T}$ proves the sequent $\chi' \vdash_{\vec{x}} \xi'$.

III.2 The classifying topos of a doctrinal site

In Section III.1, we were required at times to be unsatisfactorily vague as to which fragments of first-order logic our theories belonged, or which properties we required of our doctrines. Although imposing certain algebraic and categorical structure on our doctrines is the most intuitive method to model particular logical syntax, it is too *ad hoc* for our purposes. We would prefer instead a unified language with which we can simultaneously treat first-order theories from various underlying syntaxes.

In this section, we will observe that, in many cases, it is possible to encode further structural properties of a doctrine $P\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$, representing logical syntax, by a particular choice of Grothendieck topology on the category $C \rtimes P$. We are therefore motivated to work with *doctrinal sites*:

Definition III.15. A *doctrinal site* (also called a *fibred preorder site* in [24]) consists of a doctrine $P\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ and a Grothendieck topology J on the category $C \rtimes P$.

A doctrinal site (P, J) is equivalent in data to a relative site of the form

$$[\pi_P\colon (C \rtimes P, J) \to (C, J_{\mathrm{triv}})]$$

where π_P is a faithful fibration. Our reasons for restricting to base categories endowed with the trivial topology are discussed in Remark IV.18.

The philosophical interpretation. As aforementioned, our overarching goal in pursuing a doctrinal foundation to classifying topos theory is to yield a notion of classifying topos for as many systems of predicate reasoning as possible, without any prejudice on what the syntax of the system may be.

We already saw in Section III.1 how any formal system of predicate reasoning, with only the mildest of conditions, can be assigned a doctrine. However, we gave in Example III.12 an example of a doctrine that simultaneously represented two inequivalent theories in different fragments of first-order logic. In order to differentiate the intended syntax of a doctrine, we must therefore encode further information about the doctrine.

This is the role played by the Grothendieck topology in a doctrinal site. It is intended to capture

(i) further syntactic properties of the doctrine – such as to which fragment of first-order logic the associated theory belongs,
(ii) and information of the desired semantics of P.

The formalism of doctrinal sites is inherently flexible, being able to capture the (set-based) semantics of theories from a wide class of syntaxes.

The relative topos of a doctrinal site. Since a doctrinal site (P, J) is an example of a relative site, it is natural to contemplate the induced relative topos

$$\mathbf{Sh}(C \rtimes P, J) \to \mathbf{Sets}^{C^{\mathrm{op}}}.$$

The principal observation of this section is that the relative topos $\mathbf{Sh}(C \rtimes P, J)$ satisfies the universal property of the *classifying topos* of the doctrine – i.e. for each topos $\mathcal{F}$, there is an equivalence of categories

$$\mathbf{Geom}(\mathcal{F}, \mathbf{Sh}(C \rtimes P)) \simeq P\text{-mod}(\mathcal{F}),$$

natural in $\mathcal{F}$, where P-mod$(\mathcal{F})$ denotes the category of desired models of P in the topos $\mathcal{F}$. Hence, we will have demonstrated the final process

$$\textbf{Doctrines/Doctrinal sites} \longrightarrow \textbf{Topoi},$$

completing the schematic (III.i).

Suppose that $\mathbb{T}$ is a theory with a classifying topos. By the fact that the models of a theory $\mathbb{T}$ coincide with the models of the doctrine $F^{\mathbb{T}}$, the classifying topos of the doctrine $F^{\mathbb{T}}$ coincides with the classifying topos $\mathcal{E}_{\mathbb{T}}$ of the theory $\mathbb{T}$. A comparison with the usual construction of the classifying topos of $\mathbb{T}$, involving syntactic categories, is performed in Section III.3.

Overview. The section is divided as follows.

- At the suggestion of Kock and Reyes [73], the model of a doctrine P in a topos $\mathcal{F}$ can be described as a morphism of doctrines $P \to \mathrm{Sub}_{\mathcal{F}}$ that preserves the necessary categorical structure. We begin in Section III.2.1 by constructing the classifying topos of a primary doctrine.
- We then discuss the classifying topoi of doctrines modelling richer logical syntax in Section III.2.2. We will observe that, for many of the most widely considered doctrines, the additional syntactic structure present can be encoded by a choice of Grothendieck topology.
- Thus motivated, we define the 2-category of **DocSites** and observe that many of the notable classes of doctrines mentioned in Examples III.10 form full and faithful 2-subcategories of **DocSites**. We define the classifying topos of a doctrinal site and note its universal property.

  Finally, we discuss how, even if the syntax of a doctrine cannot be encoded by a choice of Grothendieck topology, it is possible to make a 'best approximation' that generalises the notion of the sobrification of the space of models of a propositional theory.

III.2.1 The classifying topos of a primary doctrine

We first focus on primary doctrines and describe their classifying topoi. A natural definition for the model of a primary doctrine is adapted from [73].

Definition III.16 (Definition 3.5 [73])**.** A *model* of a primary doctrine P is simply a primary doctrine morphism

$$(F,a)\colon P \longrightarrow \mathscr{P},$$

where $\mathscr{P}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{CBool}$ is the powerset doctrine.

We further define the category of models P-mod$_{\mathrm{Prim}}(\mathbf{Sets})$ of a primary doctrine P as the category $\mathbf{PrimDoc}(P,\mathscr{P})$ of primary doctrine morphisms $P \to \mathscr{P}$.

If $\mathbb{T}$ is a first-order theory involving only the symbols $\{\wedge, \top\}$, the models of $\mathbb{T}$ are easily seen to coincide with the models of the primary doctrine $F^{\mathbb{T}}$. For a model $(G,a)\colon F^{\mathbb{T}} \to \mathscr{P}$ of $F^{\mathbb{T}}$, the functor $G\colon \mathbf{Con}_N \to \mathbf{Sets}$ picks out the interpretation of each context in the model, while the meet-semilattice homomorphism $a_{\vec{x}}\colon F^{\mathbb{T}}(\vec{x}) \to \mathscr{P}(G(\vec{x}))$ sends a proposition in context $\vec{x}$ to its interpretation as a subset of $G(\vec{x})$.

Of course, we could also replace the powerset doctrine with the subobject doctrine $\mathrm{Sub}_{\mathcal{E}}\colon \mathcal{E}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ of a topos $\mathcal{E}$ to obtain a category P-mod$(\mathcal{E})$ of models of a doctrine P in any topos.

Theorem III.17. *Let $P\colon C^{\mathrm{op}} \to \mathbf{MSLat}$ be a primary doctrine. The presheaf topos $\mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}$* classifies *the doctrine P, i.e. there is a natural equivalence*

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}) \simeq P\text{-mod}_{\mathrm{Prim}}(\mathcal{E})$$

for each topos $\mathcal{E}$.

Proof. By the relative Diaconescu's equivalence Theorem I.21 and the version for **PreOrd**-valued relative sites in Corollary I.25, there is an equivalence

$$\mathbf{Topos}\begin{pmatrix} \mathcal{F} & \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}} \\ \big\downarrow f \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sets}^{C^{\mathrm{op}}} \end{pmatrix} \simeq \mathbf{RelMorph}\begin{pmatrix} (C\rtimes P, J_{\mathrm{triv}}) & (\mathcal{E}\rtimes \mathrm{Sub}_{\mathcal{F}}(f^*-), \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P \quad , & \big\downarrow \pi_{\mathcal{F}} \\ (C, J_{\mathrm{triv}}) & (\mathcal{E}, J_{\mathrm{can}}) \end{pmatrix}.$$

In particular, for each topos $\mathcal{E}$, there is an equivalence

$$\mathbf{Topos}\begin{pmatrix} \mathcal{E} & \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}} \\ \Big\| \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sets}^{C^{\mathrm{op}}} \end{pmatrix} \simeq \mathbf{RelMorph}\begin{pmatrix} (C\rtimes P, J_{\mathrm{triv}}) & (\mathcal{E}\rtimes \mathrm{Sub}_{\mathcal{E}}(-), \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P \quad , & \big\downarrow \pi_{\mathcal{E}} \\ (C, J_{\mathrm{triv}}) & (\mathcal{E}, J_{\mathrm{can}}) \end{pmatrix},$$

which is moreover natural in $\mathcal{E}$.

There is an evident equivalence

$$\mathbf{Geom}\left(\mathcal{E}, \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}\right) \simeq \mathbf{Topos}\begin{pmatrix} \mathcal{E} & \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}} \\ \Big\| \;, & \big\downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sets}^{C^{\mathrm{op}}} \end{pmatrix}.$$

The equivalence acts on objects by sending a geometric morphism $\mathcal{E} \to \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}$ to the composite

$$\mathcal{E} \longrightarrow \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}} \xrightarrow{C_{\pi_P}} \mathbf{Sets}^{C^{\mathrm{op}}}$$

while, in the converse direction, a morphism of relative topoi

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}} \arrow[d, "C_{\pi_P}"] \\ \mathcal{E} \arrow[r] & \mathbf{Sets}^{C^{\mathrm{op}}} \end{tikzcd}$$

is sent to the factoring geometric morphism $\mathcal{E} \to \mathbf{Sets}^{(C\rtimes P)^{\mathrm{op}}}$.

By Proposition I.28, the category

$$\mathbf{RelMorph}\left(\begin{array}{ccc} (\mathcal{C} \rtimes P, J_{\mathrm{triv}}) & & (\mathcal{E} \rtimes \mathrm{Sub}_{\mathcal{E}}(-), \tilde{J}_{\mathrm{can}}) \\ \big\downarrow \pi_P & , & \big\downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J_{\mathrm{triv}}) & & (\mathcal{E}, J_{\mathrm{can}}) \end{array}\right)$$

is equivalent to the category

$$\mathbf{RelMorph}_{\mathrm{cart}}((\mathcal{C}, J_{\mathrm{triv}}, P, J_{\mathrm{triv}}), (\mathcal{E}, J_{\mathrm{can}}, \mathrm{Sub}_{\mathcal{E}}, \tilde{J}_{\mathrm{can}}))$$

whose objects are pairs (F, a) of a left exact functor $F\colon \mathcal{C} \to \mathcal{E}$ and a (pseudo-)natural transformation $a\colon P \Rightarrow \mathrm{Sub}_{\mathcal{E}}$ where each component $a_c\colon P(c) \to \mathrm{Sub}_{\mathcal{E}}(F(c))$ preserves finite meets. In other words, it is equivalent to the category

$$\mathbf{PrimDoc}(P, \mathrm{Sub}_{\mathcal{E}}) = P\text{-mod}(\mathcal{E})$$

(for the equivalence on arrows, see Remark III.2). Thus, the topos $\mathbf{Sets}^{(\mathcal{C}\rtimes P)^{\mathrm{op}}}$ satisfies the universal property of the classifying topos of the primary doctrine P. □

III.2.2 Classifying topoi for richer syntaxes

What if our doctrine interprets a richer syntax? If $\mathbb{T}$ is a theory of a fragment of logic containing at least the symbols $\{\wedge, \top\}$, the models of $\mathbb{T}$ coincide with those primary doctrine morphisms $F^{\mathbb{T}} \to \mathrm{Sub}_{\mathcal{E}}$ that preserve the additional categorical structure that interprets the further necessary logical syntax. Therefore, just as with primary doctrines and following [73], we define a *model* of an existential/coherent/etc. doctrine P in a topos $\mathcal{E}$ as a morphism of existential/coherent/etc. doctrines $(F, a)\colon P \to \mathrm{Sub}_{\mathcal{E}}$. We denote the resultant categories of models by $P\text{-mod}_{\mathrm{Ex}}(\mathcal{E})$/$P\text{-mod}_{\mathrm{Coh}}(\mathcal{E})$/etc.

Encoding syntax with Grothendieck topologies. As aforementioned, we can evade this *ad hoc* approach to logical syntax, which is unsatisfactory for our desired holistic treatment, by encoding logical syntax using Grothendieck topologies. Note that, if P and Q are, say, existential doctrines, then there is a chain of inclusions

$$\mathbf{CohDoc}(P, Q) \subseteq \mathbf{ExDoc}(P, Q) \subseteq \mathbf{PrimDoc}(P, Q).$$

This suggests that the progressively more expressive syntaxes can be captured by a chain of Grothendieck topologies $J_{\mathrm{Prim}} \subseteq J_{\mathrm{Ex}} \subseteq J_{\mathrm{Coh}}$. The following proposition expresses precisely this fact (the topology J_{Prim} is simply the trivial topology on $\mathcal{C} \rtimes P$).

Proposition III.18. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to$ **MSLat** and $Q\colon \mathcal{D}^{\mathrm{op}} \to$ **MSLat** be primary doctrines, and let $(F, a)\colon P \to Q$ be a morphism of primary doctrines.*

(i) If P and Q are existential doctrines, the morphism (F, a) also preserves the existential structure if and only if the induced functor $F \rtimes a\colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$ sends J_{Ex}-covering sieves in $\mathcal{C} \rtimes P$ to J_{Ex}-covering sieves in $\mathcal{D} \rtimes Q$, where J_{Ex} is the Grothendieck topology on $\mathcal{C} \rtimes P$ (respectively, $\mathcal{D} \rtimes Q$) generated by covering families of the form

$$(d, x) \xrightarrow{g} (c, \exists_g x),$$

for each $x \in P(d)$ and $d \xrightarrow{g} c \in \mathcal{C}$ (resp., each $x \in Q(d)$ and $d \xrightarrow{g} c \in \mathcal{D}$).

(ii) *If P and Q are coherent doctrines, (F, a) is a morphism of coherent doctrines if and only if the induced functor $F \rtimes a \colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$ sends J_{Coh}-covering sieves in $\mathcal{C} \rtimes P$ to J_{Coh}-covering sieves in $\mathcal{D} \rtimes Q$, where J_{Coh} denotes the Grothendieck topology generated by covering families of the form*

$$(d, x) \xrightarrow{g} (c, \exists_g x \vee \exists_h y) \xleftarrow{h} (e, y).$$

(iii) *If P and Q are Boolean doctrines, (F, a) preserves the Boolean structure if and only if it is a morphism of coherent doctrines.*

Proof. Suppose that P and Q are existential doctrines. Recall that the morphism of primary doctrines $(F, a) \colon P \to Q$ is also a morphism of existential doctrines if and only if, for each $x \in P(d)$ and $d \xrightarrow{g} c \in \mathcal{C}$,

$$a_c \exists_{P(g)}(x) = \exists_{Q(F(g))} a_d(x).$$

This is precisely equivalent to requiring that the image under the induced functor $F \rtimes a \colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$ of the J_{Ex}-covering arrow

$$(d, x) \xrightarrow{g} (c, \exists_{P(g)} x)$$

in $\mathcal{C} \rtimes P$, i.e. the arrow

$$(F(d), a_d(x)) \xrightarrow{F(g)} (F(c), a_c \exists_{P(g)}(x)),$$

is also J_{Ex}-covering in $\mathcal{D} \rtimes Q$. This completes the proof of (i). Part (ii) is similarly demonstrated.

We turn to (iii). In one direction, every morphism of Boolean doctrines is necessarily coherent. Conversely, suppose that $(F, a) \colon P \to Q$ is a coherent doctrine morphism. We deduce that, since complements are unique (see [31, §4.13]), and we have that, for each $x \in P(c)$,

$$a_c(x \wedge \neg x) = a_c(\bot) = \bot, \quad a_c(x \vee \neg x) = a_c(\top) = \top,$$

the lattice homomorphism $a_c \colon P(c) \to Q(F(c))$ must also preserve negation and hence is a Boolean algebra homomorphism. Then, using that, for a Boolean doctrine, $\forall_{P(f)} = \neg \exists_{P(f)} \neg$ for each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, we conclude that (F, a) also preserves the interpretation of universal quantification, completing proof of (iii). □

Corollary III.19. (i) *Given an existential doctrine P, the topos $\mathbf{Sh}(\mathcal{C} \rtimes P, J_{\mathrm{Ex}})$ classifies the doctrine P.*

(ii) *Given a coherent/Boolean doctrine P, the topos $\mathbf{Sh}(\mathcal{C} \rtimes P, J_{\mathrm{Coh}})$ classifies the doctrine P.*

Proof. Let $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ be an existential doctrine. By Theorem III.17, there is an equivalence

$$\begin{aligned}\mathbf{Geom}(\mathcal{E}, \mathbf{Sets}^{(\mathcal{C} \rtimes P)^{\mathrm{op}}}) &\simeq P\text{-mod}_{\mathrm{Prim}}(\mathcal{E}),\\ &\simeq \mathbf{RelMorph}_{\mathrm{cart}}((\mathcal{C}, J_{\mathrm{triv}}, P, J_{\mathrm{triv}}), (\mathcal{E}, J_{\mathrm{can}}, \mathrm{Sub}_{\mathcal{E}}, \tilde{J}_{\mathrm{can}})).\end{aligned}$$

To obtain the existential models $P\text{-mod}_{\mathrm{Ex}}(\mathcal{E})$ of P, by Proposition III.18, it suffices to restrict the above equivalence to the subcategory of J_{Ex}-continuous morphisms of relative sites as follows

$$\begin{tikzcd} P\text{-mod}_{\mathrm{Ex}}(\mathcal{E}) \arrow[d, hook] \arrow[r, phantom, "\simeq"] & \mathbf{RelMorph}_{\mathrm{cart}}((\mathcal{C}, J_{\mathrm{triv}}, P, J_{\mathrm{Ex}}), (\mathcal{E}, J_{\mathrm{can}}, \mathrm{Sub}_{\mathcal{E}}, \tilde{J}_{\mathrm{can}})) \arrow[d, hook] \\ P\text{-mod}_{\mathrm{Prim}}(\mathcal{E}) \arrow[r, phantom, "\simeq"] & \mathbf{RelMorph}_{\mathrm{cart}}((\mathcal{C}, J_{\mathrm{triv}}, P, J_{\mathrm{triv}}), (\mathcal{E}, J_{\mathrm{can}}, \mathrm{Sub}_{\mathcal{E}}, \tilde{J}_{\mathrm{can}})). \end{tikzcd}$$

Thus, the equivalence $\mathbf{Geom}(\mathcal{E}, \mathbf{Sets}^{(\mathcal{C}\rtimes P)^{\mathrm{op}}}) \simeq P\text{-mod}_{\mathrm{Prim}}(\mathcal{E})$ also restricts to an equivalence

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C} \rtimes P, J_{\mathrm{Ex}})) \simeq P\text{-mod}_{\mathrm{Ex}}(\mathcal{E}).$$

Hence, $\mathbf{Sh}(\mathcal{C}\rtimes P, J_{\mathrm{Ex}})$ classifies the existential doctrine P. The proof for coherent/Boolean doctrines is near identical. □

Remark III.20. After defining them in Examples III.10(iv), Heyting doctrines have been neglected in our subsequent discussion. This is because intuitionistic logic resists a unified treatment using Grothendieck topologies. Namely, the behaviour of Heyting implication $\to$ cannot, in general, be captured by an *exactness property* as enforced by a Grothendieck topology.

Despite this, if we focus exclusively on set-based models – or more generally models in Boolean topoi, then it becomes possible encode the structure of intuitionistic logic using a Grothendieck topology in the sense that, for a Heyting doctrine P, there exists a Grothendieck topology J_{Heyt} such that a morphism of primary doctrines

$$(F, a) \colon P \longrightarrow \mathscr{P}$$

is a morphism of Heyting doctrines if and only if the induced functor

$$F \rtimes a \colon \mathcal{C} \rtimes P \longrightarrow \mathbf{Sets} \rtimes \mathscr{P}$$

sends J_{Heyt}-covering sieves to $\tilde{J}_{\mathrm{can}}$-covering sieves.

Let P be a Heyting doctrine and let $(F, a) \colon P \to \mathscr{P}$ be a model of P as an existential doctrine. The model (F, a) is also a model of P as a Heyting doctrine if and only if, for each $c \in \mathcal{C}$, the lattice homomorphism $a_c \colon P(c) \to \mathscr{P}(F(c))$ also preserves the Heyting implication $\to$. Since $\mathscr{P}(F(c))$ is a Boolean algebra, this is equivalent to requiring that, for each $x, y \in P(c)$,

$$\neg a_c(x) \vee a_c(y) = a_c(x) \to a_c(y) = a_c(x \to y).$$

In particular, as complements in $\mathscr{P}(F(c))$ are unique, a_c preserves *pseudo-complements* (i.e. Heyting negation $- \to \bot$) if and only if, for each $x \in P(c)$,

$$a_c(\neg x \vee x) = a_c(\neg x) \vee a_c(x) = \top = a_c(\top)$$

Thus, we deduce that (F, a) is a Heyting model of P if and only if the induced functor $F \rtimes a \colon \mathcal{C} \rtimes P \to \mathbf{Sets} \rtimes \mathscr{P}$ is J_{Heyt}-continuous, where J_{Heyt} is the Grothendieck topology on $\mathcal{C} \rtimes P$ generated by following three species of covering families:

(i) firstly, families of the form

$$(d, x) \xrightarrow{g} (c, \exists_g x \vee \exists_h y) \xleftarrow{h} (e, y),$$

for $x \in P(d)$, $y \in P(e)$, and arrows $d \xrightarrow{g} c, e \xrightarrow{h} c \in C$ (ensuring that (F, a) preserves the coherent structure of P);

(ii) secondly, families of the form

$$(c, \neg x) \xrightarrow{\mathrm{id}_c} (c, \top) \xleftarrow{\mathrm{id}_c} (c, x),$$

for all $c \in C$ and $x \in P(c)$ (ensuring that a_c preserves pseudo-complements);

(iii) and finally, families of the form

$$(c, \neg x) \xrightarrow{\mathrm{id}_c} (c, x \to y) \xleftarrow{\mathrm{id}_c} (c, y),$$

for all $c \in C$ and $x, y \in P(c)$ (ensuring that a_c preserves Heyting implication).

Requiring that a morphism $(F, a) \colon P \to \mathscr{P}$, as primary doctrines, is J_{Heyt}-continuous is equivalent to requiring that (F, a) factors through the *Boolean completion* of the Heyting doctrine P from [46, §2.3]. However, if we wish to consider the semantics of an intuitionistic theory in non-Boolean topoi, we cannot expect to find a Grothendieck topology that encodes the semantics of the theory, as evidenced in Example III.21 below.

Example III.21. We give a simple example of a Heyting doctrine whose models in a pair of topoi cannot simultaneously be captured by the choice of a single Grothendieck topology. For simplicity, we will consider a Heyting algebra H (i.e. the Heyting doctrine for a propositional intuitionistic theory) since a morphism of primary doctrines $H \to \mathrm{Sub}_{\mathcal{E}}$ is equivalent in datum to a finite meet preserving map $H \to \mathrm{Sub}_{\mathcal{E}}(1)$.

Let $\mathbf{3}$ denote the 3-element Heyting algebra

$$\begin{tikzcd} \top \arrow[d, dash] \\ \# \arrow[d, dash] \\ \bot. \end{tikzcd}$$

Equivalently, $\mathbf{3}$ is the frame of opens for the Sierpinski space $\mathbb{S}$. As a Heyting algebra, there is a unique $\mathbf{2}$-valued model of $\mathbf{3}$, the homomorphism of Heyting algebras

$$\begin{tikzcd} \top \arrow[d, dash] \arrow[dr, bend left] & \\ \# \arrow[d, dash] \arrow[r, bend right] & \top \arrow[d, dash] \\ \bot \arrow[r, bend right] & \bot, \end{tikzcd}$$

which corresponds to the open point of $\mathbb{S}$. It is the unique finite meet preserving map $f\colon \mathbf{3} \to \mathbf{2}$ for which both the sieve $\{\,\bot \leqslant \top, \# \leqslant \top\,\}$ on $\top$ and the empty sieve $\emptyset$ on $\bot$ are sent by f to J_{can}-covering sieves in $\mathbf{2}$ (here J_{can} denotes the canonical topology on the frame $\mathbf{2}$).

Therefore, the category of models of $\mathbf{3}$ as a Heyting doctrine in the topos of sets $\mathbf{Sets} \simeq \mathbf{Sh}(\mathbf{2})$ can be encoded by a Grothendieck topology $J_{\# \to \top}$ in the sense that there is an equivalence

$$\mathbf{1} \simeq \mathbf{HeytDoc}(\mathbf{3}, \mathbf{2}) = \mathbf{3}\text{-mod}_{\mathrm{Heyt}}(\mathbf{Sets}) \simeq \mathbf{Geom}(\mathbf{Sets}, \mathbf{Sh}(\mathbf{3}, J_{\# \to \top})),$$

where $J_{\# \to \top}$ is the Grothendieck topology whose covering sieves are:

(i) $J_{\# \to \top}(\top) = \Big\{\, \{\,\bot \leqslant \top, \# \leqslant \top\,\},\ \{\,\bot \leqslant \top, \# \leqslant \top, \top \leqslant \top\,\}\,\Big\}$,

(ii) $J_{\# \to \top}(\#) = \Big\{ \{\,\bot \leqslant \#, \# \leqslant \#\,\} \Big\}$,

(iii) and $J_{\# \to \top}(\bot) = \Big\{ \emptyset,\ \{\,\bot \leqslant \bot\,\} \Big\}$.

This is precisely the topology J_{Heyt} suggested in Remark III.20 above.

However, there are two $\mathbf{3}$-valued models of $\mathbf{3}$, the Heyting algebra homomorphisms

$$\begin{tikzcd} \top \arrow[r] \arrow[d, dash] & \top \arrow[d, dash] \\ \# \arrow[r] \arrow[d, dash] & \# \arrow[d, dash] \\ \bot \arrow[r] & \bot, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \top \arrow[r] \arrow[d, dash] & \top \arrow[d, dash] \\ \# \arrow[ur, bend right] \arrow[d, dash] & \# \arrow[d, dash] \\ \bot \arrow[r] & \bot \end{tikzcd}$$

– corresponding to the two open continuous endomorphisms of $\mathbb{S}$. Of these two maps, only the latter sends $J_{\# \to \top}$-covering sieves to J_{can}-covering sieves in $\mathbf{3}$. Hence, there is no longer an equivalence

$$\mathbf{2} \simeq \mathbf{HeytDoc}(\mathbf{3}, \mathbf{3}) = \mathbf{3}\text{-mod}_{\mathrm{Heyt}}(\mathbf{Sh}(\mathbf{3})) \not\simeq \mathbf{Geom}(\mathbf{Sh}(\mathbf{3}), \mathbf{Sh}(\mathbf{3}, J_{\# \to \top})) \simeq \mathbf{1}.$$

In other words, we cannot simultaneously describe the models of $\mathbf{3}$ (as a Heyting algebra) in arbitrary topoi using a single Grothendieck topology on $\mathbf{3}$.

III.2.3 The 2-category of doctrinal sites.

Our use of doctrinal sites as a formalism is motivated by a desire to exploit the unified treatment of doctrines and syntax afforded by assigning a Grothendieck topology. We now describe the 2-category of doctrinal sites, and observe, as well, that taking the *classifying topos* of a doctrinal site is bifunctorial.

Definitions III.22. (i) Given two categories $\mathcal{C}, \mathcal{D}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *flat* if it defines a morphism of sites

$$F\colon (\mathcal{C}, J_{\mathrm{triv}}) \longrightarrow (\mathcal{D}, J_{\mathrm{triv}})$$

when both $\mathcal{C}$ and $\mathcal{D}$ are endowed with the trivial topology. We note that this generalises the definition of flat as synonymous with left exact we have used previously – indeed, a flat functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves any finite limits that exist in $\mathcal{C}$ (see [107, Corollary 4.14] or Remark I.4).

(ii) We denote the *2-category of doctrinal sites* by **DocSites**.

a) The objects are doctrinal sites.

b) An arrow $(P, J) \to (Q, K)$ of **DocSites** is a morphism of the associated relative sites, i.e. a pair (F, a) consisting of a *flat* functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $a\colon P \Rightarrow Q \circ F^{\mathrm{op}}$ such that the induced functor

$$F \rtimes a\colon (\mathcal{C} \rtimes P, J) \longrightarrow (\mathcal{D} \rtimes Q, K)$$

is a morphism of sites.

c) A 2-cell between two morphisms of doctrinal sites

$$\begin{tikzcd} (P,J) \arrow[rr, bend left, "{(F,a)}"] \arrow[rr, bend right, "{(F',a')}"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & (Q,K) \end{tikzcd}$$

is a natural transformation $\alpha\colon F \Rightarrow F'$ such that

$$a_c(x) \leqslant Q(\alpha_c)(a'_c(x))$$

for each object $c \in \mathcal{C}$ and $x \in P(c)$.

Example III.23. Although the definition of a morphism of doctrinal sites may appear initially unmotivated within the context of standard doctrine theory, we note that, for many of the examples of doctrines that we have encountered, the definition coincides with the pre-existing notions we have for morphisms of doctrines.

If $P\colon \mathcal{C}^{\mathrm{op}} \to$ **MSLat** and $Q\colon \mathcal{D}^{\mathrm{op}} \to$ **MSLat** are both primary doctrines, then a morphism $(F, a)\colon (P, J) \to (Q, K) \in$ **DocSites** of doctrinal sites, where K is a relatively subcanonical topology, is a morphism of primary doctrines in the sense of Examples III.10(i) such that, in addition,

$$F \rtimes a\colon (\mathcal{C} \rtimes P, J) \longrightarrow (\mathcal{D} \rtimes Q, K)$$

is cover preserving. In particular, the morphisms $(P, J_{\mathrm{triv}}) \to (Q, J_{\mathrm{triv}})$ of **DocSites** are precisely morphisms of primary doctrines. Thus, there exists a full and faithful 2-embedding

$$\mathbf{PrimDoc} \hookrightarrow \mathbf{DocSites}$$

that sends a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to$ **MSLat** to the doctrinal site (P, J_{triv}).

Similarly, by Proposition III.18, there exist full and faithful embeddings of 2-subcategories

$$\begin{tikzcd} \mathbf{ExDoc} \arrow[dr, hook] & \\ \mathbf{CohDoc} \arrow[r, hook] & \mathbf{DocSites} \\ \mathbf{BoolDoc} \arrow[ur, hook] & \end{tikzcd}$$

which send an existential (respectively, coherent/Boolean) doctrine

$$P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{MSLat}$$

to the doctrinal site (P, J_{Ex}) (respectively, (P, J_{Coh})).

Definition III.24. For a doctrinal site (P, J), we will call the relative topos

$$\mathbf{Sh}(\mathcal{C} \rtimes P, J) \longrightarrow \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

the *classifying topos* of (P, J).

This is the classifying topos of a doctrinal site (P, J) in the sense that, for any topos $\mathcal{E}$, by Corollary I.25 and in a similar fashion to Theorem III.17 and Corollary III.19, there is an equivalence

$$\begin{aligned}
\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C} \rtimes P, J)) &\simeq \mathbf{Topos}\left(\begin{array}{ccc} \mathcal{E} & & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \Vert & , & \big\downarrow{\scriptstyle C_{\pi_P}} \\ \mathcal{E} & & \mathbf{Sh}(\mathcal{C}, J) \end{array} \right), \\
&\simeq \mathbf{RelMorph}\left(\begin{array}{ccc} (\mathcal{C} \rtimes P, K) & & (\mathcal{E} \rtimes \mathrm{Sub}_{\mathcal{E}}(-), \tilde{J}_{\mathrm{can}}) \\ \big\downarrow{\scriptstyle \pi_P} & , & \big\downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ (\mathcal{C}, J) & & (\mathcal{E}, J_{\mathrm{can}}) \end{array} \right), \\
&\simeq \mathbf{DocSites}((P, J), (\mathrm{Sub}_{\mathcal{E}}, \tilde{J}_{\mathrm{can}})).
\end{aligned}$$

Since doctrinal sites are examples of relative sites, morphisms of doctrinal sites are morphisms of relative sites (see Definition I.18), and the 2-cells of **DocSites** induce 2-cells between morphisms of relative sites Remark III.2, there is an evident bifunctor

$$\mathbf{DocSites} \longrightarrow \mathbf{RelTopos}.$$

The map on objects $\mathbf{RelTopos} \to \mathbf{Topos}$ that sends a relative topos $\mathcal{F} \to \mathcal{E}$ to the domain topos $\mathcal{F}$ is also bifunctorial, and the composite

$$\mathbf{DocSites} \longrightarrow \mathbf{RelTopos} \longrightarrow \mathbf{Topos}$$

is the bifunctor that sends a doctrinal site to its classifying topos.

This completes the last process **Doctrinal sites** $\to$ **Topoi** in schematic (III.i). Since the models of a theory $\mathbb{T}$, in a suitable syntax, coincide with the models of the doctrine $F^{\mathbb{T}}$, the triangle of processes

$$\begin{tikzcd}
\begin{array}{c}\textbf{Theories} \\ \textbf{+ syntax}\end{array} \arrow[rr, "\mathbb{T} \mapsto (F^{\mathbb{T}}, K)"] \arrow[dr, bend right, "\mathbb{T} \mapsto \mathcal{E}_{\mathbb{T}}"'] & & \begin{array}{c}\textbf{Doctrinal} \\ \textbf{sites}\end{array} \arrow[dl, bend left, "(P,J) \mapsto \mathbf{Sh}(\mathcal{C} \rtimes P, J)"] \\
& \textbf{Topoi} &
\end{tikzcd}$$

commutes in that $\mathbf{Sh}(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K) \simeq \mathcal{E}_{\mathbb{T}}$, where K is the appropriate Grothendieck topology on $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$ corresponding to the underlying syntax of the theory $\mathbb{T}$.

Sobrifying the desired models

Definition III.25. Following the above discussion, the following are equivalent:

(i) The syntactic rules of a theory concern *exactness* properties, by which we mean that the syntactic rules can be encoded by a Grothendieck topology as above;

(ii) The theory has a classifying topos.

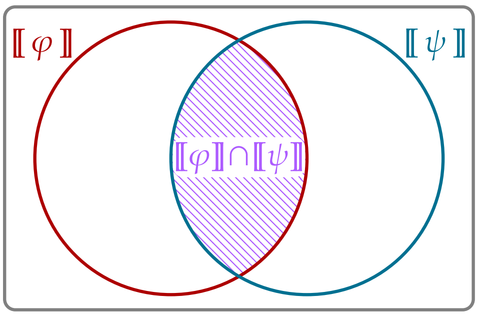
Since such a theory is therefore Morita equivalent to a geometric theory, we might call such theories *geometrizable*.

Even when a theory is not geometrizable, there is a 'best approximation' by a geometrizable theory that generalises the notion of the *sobrification* of the space of models of a propositional theory.

Sobrifying the space of models of a propositional theory. Let $\mathbb{T}$ be a propositional theory over a signature Σ, and let $\mathbb{T}$-mod($\mathbf{2}$) denote the set of $\mathbf{2}$-valued models of $\mathbb{T}$. Each formula φ over Σ defines a subset of $\mathbb{T}$-mod($\mathbf{2}$) –

$$[\![\varphi]\!] = \{ M \in \mathbb{T}\text{-mod}(\mathbf{2}) \mid M \vDash \varphi \},$$

the set of models that satisfy φ, otherwise called the *interpretation* of φ. Using set-theoretic operations, the definable subsets can be manipulated in a manner consistent with the logical operations of intuitionistic logic, e.g. the interpretation of the conjunction of two formulae is expressed by the intersection as in the diagram



$\mathbb{T}$-mod($\mathbf{2}$).

With these subsets as the basic opens, it is possible to generate a topology on the set $\mathbb{T}$-mod($\mathbf{2}$). We call the resultant topological space the *space of models*.

The frame of opens $\mathcal{O}(\mathbb{T}\text{-mod}(\mathbf{2}))$ is the Lindenbaum-Tarski algebra for a geometric propositional theory $\mathbb{T}'$. Hence, if $\mathbb{T}$-mod($\mathbf{2}$) is a sober space, then

$$\mathbb{T}\text{-mod}(\mathbf{2}) \simeq \mathbf{Loc}(\mathbf{2}, \mathcal{O}(\mathbb{T}\text{-mod}(\mathbf{2}))) \simeq \mathbb{T}'\text{-mod}(\mathbf{2}),$$

i.e. if $\mathbb{T}$-mod($\mathbf{2}$) is sober, then $\mathbb{T}$ is geometrizable. Even when $\mathbb{T}$ is not geometrizable, the sobrification of the space $\mathbb{T}$-mod($\mathbf{2}$) evidently yields the 'best approximation' of $\mathbb{T}$ by a geometrizable theory.

Example III.26 (Cofinite naturals)**.** What would an *ungeometrizable* propositional theory look like? Let $\mathbb{N}^{\text{cof}}$ denote the space of the natural numbers endowed with the cofinite topology. The frame of opens $\mathcal{O}(\mathbb{N}^{\text{cof}})$ is the Lindenbaum-Tarski algebra for the following geometric propositional theory $\mathbb{T}_{\mathbb{N}^{\text{cof}}}$.

(i) For each natural number $n \in \mathbb{N}$, there is a basic proposition $[x \neq n]$.

(ii) The axioms of $\mathbb{T}_{\mathbb{N}^{\mathrm{cof}}}$ consist of the sequents

$$\top \vdash [x \neq n] \vee [x \neq n']$$

for each pair $n, n' \in \mathbb{N}$ with $n \neq n'$.

For each $n \in \mathbb{N}$, there is a **2**-valued model p_n of $\mathbb{T}_{\mathbb{N}^{\mathrm{cof}}}$ that evaluates basic propositions as

$$p_n([x \neq n']) = \begin{cases} \top & \text{if } n \neq n', \\ \bot & \text{if } n = n'. \end{cases}$$

But there is a further **2**-valued model p_∞ 'at infinity' where $p_\infty([x \neq n]) = \top$ for all $n \in \mathbb{N}$. In other words, there is an extra point in the sobrification $\mathrm{Pt}\mathcal{O}(\mathbb{N}^{\mathrm{cof}})$ of $\mathbb{N}^{\mathrm{cof}}$.

The only model of $\mathbb{T}_{\mathbb{N}^{\mathrm{cof}}}$ that satisfies the infinite conjunction $\bigwedge_{n \in \mathbb{N}} [x \neq n]$ is p_∞. Therefore, the models for the theory

$$\mathbb{T}' = \mathbb{T}_{\mathbb{N}^{\mathrm{cof}}} \cup \left\{ \bigwedge_{n \in \mathbb{N}} [x \neq n] \vdash \bot \right\}$$

correspond to the points of the inebriated[1] space $\mathbb{N}^{\mathrm{cof}} \subseteq \mathrm{Pt}\mathcal{O}(\mathbb{N}^{\mathrm{cof}})$.

However, we cannot simply consider $\mathbb{T}'$ as a theory over the syntax of infinitary propositional logic since then we would introduce new definable subsets to the space of models $\mathbb{T}'\text{-mod}(\mathbf{Sets}) \simeq \mathbb{N}$, and therefore change the topology. Instead, we must make the unnatural restriction that the only well-formed formula involving an infinite conjunction is the formula $\bigwedge_{n \in \mathbb{N}} [x \neq n]$.

The desired models of a primary doctrine. The same intuition observed in the propositional case extends to the predicate setting. Let $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ be a primary doctrine representing a predicate theory that interprets the symbols $\{\wedge, \top\}$. By the *desired models* of P, we mean a specified subcategory

$$P\text{-mod}(\mathbf{Sets}) \subseteq \mathbf{PrimDoc}(P, \mathscr{P}).$$

The category P-mod(**Sets**) represents those set-based models of P as a primary doctrine that also preserve unspecified further syntactic structure. We might be tempted to make natural assumptions about P-mod(**Sets**), such as that it ought to be a full subcategory or *replete* under isomorphisms, but these will prove unnecessary.

The pair $(P, P\text{-mod}(\mathbf{Sets}))$ of a doctrine and a category of its desired set-based models is said to be *geometrizable* if P-mod(**Sets**) is of the form

$$P\text{-mod}(\mathbf{Sets}) \simeq \mathbf{DocSites}((P, J), (\mathscr{P}, \tilde{J}_{\mathrm{can}})) \subseteq \mathbf{PrimDoc}(P, \mathscr{P})$$

for some Grothendieck topology J on $\mathcal{C} \rtimes P$. In other words, the syntactic rules that specify our desired models P-mod(**Sets**) can be encoded by a Grothendieck topology. When this is the case, the doctrine P has a classifying topos (for set-based models) since, just as in Corollary III.19, there is an equivalence

$$P\text{-mod}(\mathbf{Sets}) \simeq \mathbf{DocSites}((P, J), (\mathscr{P}, \tilde{J}_{\mathrm{can}})) \simeq \mathbf{Geom}(\mathbf{Sets}, \mathbf{Sh}(\mathcal{C} \rtimes P, J)).$$

[1]That is to say, not sober.

Evidently, a 'best approximation' to P-mod(**Sets**) by a category of the form

$$\mathbf{DocSites}((P,J),(\mathscr{P},\tilde{J}_{\mathrm{can}}))$$

can be obtained by setting J as the Grothendieck topology where a sieve

$$\left\{ (c_i,x_i) \xrightarrow{f_i} (d,y) \,\middle|\, i \in I \right\}$$

in $\mathcal{C} \rtimes P$ is J-covering if and only if, for every desired model $(F,a) \in P$-mod(**Sets**), the image

$$\left\{ a_{c_i}(x_i) \xrightarrow{F(f_i)|_{a_{c_i}(x_i)}} a_d(y) \,\middle|\, i \in I \right\}$$

is jointly surjective (i.e. $F \rtimes a$ sends J-covers to $\tilde{J}_{\mathrm{can}}$-covers).

This is evidently the 'best approximation' of the pair $(P, P\text{-mod}(\mathbf{Sets}))$ by a geometrizable pair $(P, \mathbf{DocSites}((P,J),(\mathscr{P},\tilde{J}_{\mathrm{can}})))$ in that it describes an adjunction

$$\begin{tikzcd} \mathbf{DesiredModels}(P) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Geomtrizable}(P) \arrow[l, hook, shift left] \end{tikzcd}$$

where **DesiredModels**(P) is the poset of subcategories of **PrimDoc**($P,\mathscr{P}$) ordered by inclusion, and **Geomtrizable**(P) is the subposet of geometrizable categories of desired models.

III.3 Syntactic categories

The standard textbook accounts of classifying topos theory [87], [22, §2], [63, §D3], [79, §X] construct the topos $\mathcal{E}_{\mathbb{T}}$ using the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of a theory $\mathbb{T}$. In order to express the syntactic category construction, the theory $\mathbb{T}$ must exist in a fragment of first-order logic that interprets regular logic (i.e. the fragment whose permissible symbols are $\{\wedge, \top, \exists, =\}$). Thus, as previously emphasised, our framework of doctrinal sites enables the construction of classifying topoi for theories from weaker logical syntaxes.

When the necessary regular structure is present, the two approaches, i.e. using doctrines and syntactic categories to represent classifying topoi, evidently yield equivalent topoi by the universal property of the classifying topos. In this section, we will compare the two approaches in more detail.

Overview. We proceed as follows.

- We first recall the necessary background on *existential sites* from [24], in which language we phrase our development.
- An *existential doctrinal site* whose underlying doctrine is also a primary doctrine has enough expressive power to construct a 'syntactic category'. We recall this construction in Section III.3.2, as well as the pseudo-adjunction **Syn** $\dashv \mathrm{Sub}_{(-)}$ between the syntactic category construction and taking the doctrine of subobjects.
- In Section III.3.3, the pseudo-adjunction **Syn** $\dashv \mathrm{Sub}_{(-)}$ is extended to give an pseudo-adjunction between existential doctrinal sites and regular sites.

– The two topoi of sheaves we can now associate with an existential doctrinal site – $\mathbf{Sh}(C \rtimes P, J_{\rtimes})$ and $\mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}})$ – are compared in Section III.3.4. For an existential doctrine P, we exhibit a functor $\zeta^P : C \rtimes P \to \mathbf{Syn}(P)$ that yields a dense morphism of sites

$$\zeta^P : (C \rtimes P, J_{\rtimes}) \longrightarrow (\mathbf{Syn}(P), J_{\mathbf{Syn}})$$

and hence an equivalence $\mathbf{Sh}(C \rtimes P, J_{\rtimes}) \simeq \mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}})$.

III.3.1 Existential doctrinal sites

First, we recall some definitions from the theory of *existential fibred sites* [24, §5]. Note that the exposition in [24] exists in the more general framework of indexed categories $F : C^{\mathrm{op}} \to \mathfrak{CAT}$, whereas we have elected to restrict to indexed preorders (or doctrines in our language).

Definition III.27 (Definition 5.1 [24])**.** Let $P : C^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine such that, for each arrow $d \xrightarrow{f} c \in C$, the map $P(f) : P(c) \to P(d)$ has a left adjoint $\exists_f$. Suppose we are also given, for each $c \in C$, a Grothendieck topology J_c on the preorder $P(c)$ for which $\exists_f : P(c) \to P(d)$ sends J_c-covers to J_d-covers.

(i) We say that the pair $(P, (J_c)_{c\in C})$ satisfies the *relative Frobenius condition* if for each sieve S on the object $(d, y) \in C \rtimes P$ for which the sieve

$$\left\{ \exists_f z \leqslant y \,\middle|\, (c, z) \xrightarrow{f} (d, y) \in S \right\}$$

is J_d-covering then the sieve

$$\left\{ \exists_f z \leqslant x \,\middle|\, (c, z) \xrightarrow{f} (d, y) \in S,\ z \leqslant P(f)(x) \right\}$$

is J_d-covering too, for any $x \in P(d)$ with $x \leqslant y$.

(ii) We say that the pair $(P, (J_c)_{c\in C})$ satisfies the *relative Beck-Chevalley condition* if for each sieve S on $(d, y) \in C \rtimes P$ for which the sieve

$$\left\{ \exists_f z \leqslant y \,\middle|\, (c, z) \xrightarrow{f} (d, y) \in S \right\}$$

is J_d-covering, given an arrow $e \xrightarrow{h} d \in C$, the sieve

$$\left\{ \exists_g z \leqslant P(h)(y) \,\middle|\, (c, z) \xrightarrow{f} (d, y) \in S,\ h \circ g = f \right\}$$

is J_d-covering too.

The pair $(P, (J_c)_{c\in C})$ is said to be a *existential doctrinal site* if the relative Frobenius and relative Beck-Chevalley conditions are both satisfied.

Let $P\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine such that each fibre $P(c)$ has a Grothendieck topology J_c and, for each arrow $d \xrightarrow{f} c \in C$, the map $P(f)\colon P(c) \to P(d)$ has a cover-preserving left adjoint $\exists_f$. In light of Proposition III.18, we would want to define a Grothendieck topology $J_{\rtimes}$ on the category $C \rtimes P$ for which a sieve

$$\left\{ (c_i, x_i) \xrightarrow{f_i} (d, y) \,\middle|\, i \in I \right\}$$

is $J_{\rtimes}$-covering if and only if

$$\left\{ (d, \exists_{f_i} x_i) \xrightarrow{\mathrm{id}_d} (d, y) \,\middle|\, i \in I \right\}$$

is J_d-covering. Such an assignment of sieves to objects is reflexive and transitive, but not necessarily stable. We observe that the stability of $J_{\rtimes}$ under arrows of the form $(d, x) \xrightarrow{\mathrm{id}_d} (d, y)$ (respectively, $(e, P(h)(y)) \xrightarrow{h} (d, y)$) is precisely given by the relative Frobenius condition (resp., the relative Beck-Chevalley condition), and since any arrow $(e, x) \xrightarrow{h} (d, y) \in C \rtimes P$ can be factored as

$$(e, x) \xrightarrow{\mathrm{id}_e} (e, P(h)(y)) \xrightarrow{h} (d, y)$$

we obtain the following proposition.

Proposition III.28 (Theorem 5.1 [24]). *For the doctrine $P\colon C^{\mathrm{op}} \to$ **PreOrd** above, $J_{\rtimes}$ defines a Grothendieck topology on the category $C \rtimes P$ if and only if the pair $(P, (J_c)_{c\in C})$ satisfies both the relative Frobenius and relative Beck-Chevalley conditions.*

Definition III.29 (Theorem 5.1 [24]). Let $(P, (J_c)_{c\in C})$ be an existential doctrinal site. We call the Grothendieck topology $J_{\rtimes}$ on $C \rtimes P$, where a sieve

$$\left\{ (c_i, x_i) \xrightarrow{f_i} (d, y) \,\middle|\, i \in I \right\}$$

is $J_{\rtimes}$-covering if and only if

$$\left\{ (d, \exists_{f_i} x_i) \xrightarrow{\mathrm{id}_d} (d, y) \,\middle|\, i \in I \right\}$$

is J_d-covering, the *existential topology* for the pair $(P, (J_c)_{c\in C})$.

The relative Frobenius and relative Beck-Chevalley conditions are related to the usual Frobenius and Beck-Chevalley conditions by the following proposition.

Proposition III.30 (Proposition 5.3 [24]). *Let $(P, (J_c)_{c\in C})$ be an existential doctrinal site.*

(i) If $P(c)$ has all finite meets for each $c \in C$, then the pair $(P, (J_c)_{c\in C})$ satisfies the relative Frobenius condition if and only if P satisfies the Frobenius condition - i.e.

$$\exists_f z \wedge x = \exists_f (z \wedge P(f)(x)).$$

(ii) *If C has all pullbacks, then the pair $(P, (J_c)_{c\in C})$ satisfies the relative Beck-Chevalley condition if and only if P satisfies the Beck-Chevalley condition - i.e. for each pullback square*

$$\begin{tikzcd} c \times_e d \arrow[r, "g"] \arrow[d, "k"'] & d \arrow[d, "h"] \\ c \arrow[r, "f"] & e \end{tikzcd}$$

of C, the square

$$\begin{tikzcd} P(c \times_e d) \arrow[r, "\exists_g"] & P(d) \\ P(c) \arrow[u, "P(k)"] \arrow[r, "\exists_f"] & P(e) \arrow[u, "P(h)"'] \end{tikzcd}$$

commutes.

Hence we note that, for any existential doctrinal site $(P, (J_c)_{c\in C})$, if P is a primary doctrine, then P is automatically an existential doctrine, and the existential topology $J_{\rtimes}$ contains the topology J_{Ex}. Such an existential doctrinal site should be understood as a doctrine which interprets at least regular logic, if not further, richer syntax.

Examples III.31. Let $P\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine such that, for each $d \xrightarrow{f} c \in C$, the map $P(f)\colon P(c) \to P(d)$ has a left adjoint $\exists_f$. Of particular interest is the case where the Grothendieck topology J_c assigned to each fibre $P(c)$ of an existential doctrinal site $(P, (J_c)_{c\in C})$ is subcanonical, i.e. a family $\{x_i \leqslant y \mid i \in I\}$ of inequalities in $P(c)$ is J_c-covering only if the join $\bigvee_{i\in I} x_i$ exists and $y \leqslant \bigvee_{i\in I} x_i$ (if $P(c)$ is a poset, $y = \bigvee_{i\in I} x_i$). For each arrow $d \xrightarrow{f} c$ of C, being a left adjoint, $\exists_f$ preserves all joins that exist in $P(d)$ and so $\exists_f$ is automatically cover-preserving if J_d and J_c are both subcanonical. Consideration of certain cases will allow us to generalise existential doctrines and coherent doctrines to non-cartesian base categories, as mentioned in Remark III.11.

(i) There exists an existential doctrinal site $(P, (J_{\mathrm{triv}})_{c\in C})$, where each fibre $P(c)$ has been given the trivial topology J_{triv}, if and only if:

   a) *(relative Frobenius condition)* for each arrow $d \xrightarrow{f} c$ of C, $x, y \in P(c)$ with $x \leqslant y$ and $z \in P(d)$ such that
   $$\exists_f z \leqslant y \text{ and } y \leqslant \exists_f z,$$
   there exists some $w \in P(d)$ with $w \leqslant z$ such that
   $$\exists_f w \leqslant x \text{ and } x \leqslant \exists_f w;$$

   b) *(relative Beck-Chevalley condition)* for each pair of arrows
   $$\begin{tikzcd} & e \arrow[d, "h"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$
   of C, $y \in P(c)$ and $z \in P(d)$ such that
   $$\exists_f z \leqslant y \text{ and } y \leqslant \exists_f z,$$

there exists a commutative square

$$\begin{tikzcd} e' \arrow[r, "g"] \arrow[d, "k"'] & e \arrow[d, "h"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$

in $\mathcal{C}$ such that

$$\exists_g P(k)(z) \leqslant x \text{ and } x \leqslant \exists_g P(k)(z).$$

Note that if $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ is an existential doctrine, $(P, (J_{\mathrm{triv}})_{c\in\mathcal{C}})$ is an existential doctrinal site.

We denote by J_{Ex} the existential topology on $\mathcal{C} \rtimes P$ induced by the existential doctrinal site $(P, (J_{\mathrm{triv}})_{c\in\mathcal{C}})$, i.e. the Grothendieck topology generated by covering families of the form

$$(d, x) \xrightarrow{g} (c, \exists_g x),$$

for $x \in P(d)$ and $d \xrightarrow{g} c \in \mathcal{C}$, in analogy with Proposition III.18(i). We define **RelExDoc**, the 2-category of *relative existential doctrines*, as the 1-full 2-subcategory of **DocSites** on objects of the form (P, J_{Ex}), where $(P, (J_{\mathrm{triv}})_{c\in\mathcal{C}})$ is an existential doctrinal site. The category **ExDoc** of existential doctrines is thus now a 1-full 2-subcategory of **RelExDoc**.

(ii) Suppose that we can endow each fibre $P(c)$ with the Grothendieck topology J_{Coh}, where a sieve S on $y \in P(c)$ is J_{Coh}-covering precisely if S contains a finite family

$$\{\, x_i \leqslant y \mid i \in I \,\} \subseteq S$$

such that $y \leqslant \bigvee_{i\in I} x_i$. If $P(c)$ is a lattice, J_{Coh} defines a Grothendieck topology on $P(c)$ if and only if $P(c)$ is a distributive lattice. The pair $(P, (J_{\mathrm{Coh}})_{c\in\mathcal{C}})$ defines an existential doctrinal site if and only if:

a) *(relative Frobenius condition)* for each pair $x, y \in P(c)$ with $x \leqslant y$, and each finite collection of pairs $d_i \xrightarrow{f_i} c$ and $z \in P(d_i)$, indexed by $i \in I$, such that

$$\bigvee_{i\in I} \exists_{f_i} z_i \leqslant y \text{ and } y \leqslant \bigvee_{i\in I} \exists_{f_i} z_i,$$

for each $i \in I$ there exists some $w_i \in P(d_i)$ with $w_i \leqslant z_i$ such that

$$\bigvee_{i\in I} \exists_{f_i} w_i \leqslant x \text{ and } x \leqslant \bigvee_{i\in I} \exists_{f_i} w_i;$$

b) *(relative Beck-Chevalley condition)* for each arrow $e \xrightarrow{h} c$ of $\mathcal{C}$, $y \in P(c)$, and each finite collection of pairs $d_i \xrightarrow{f_i} c$ and $z \in P(d_i)$, indexed by $i \in I$, such that

$$\bigvee_{i\in I} \exists_{f_i} z_i \leqslant y \text{ and } y \leqslant \bigvee_{i\in I} \exists_{f_i} z_i,$$

for each $i \in I$, there exists a finite collection of pairs of arrows g_j and k_j, indexed by $j \in J_i$, such that there is a commutative square

$$\begin{tikzcd} e'_j \arrow[r, "g_j"] \arrow[d, "k_j"'] & e \arrow[d, "h"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$

of $\mathcal{C}$, and secondly

$$\bigvee_{i \in I} \bigvee_{j \in J_i} \exists_{g_j} P(k_j)(z_i) \leqslant x \text{ and } x \leqslant \bigvee_{i \in I} \bigvee_{j \in J_i} \exists_{g_j} P(k_j)(z_i).$$

Just as above, by analogy with Proposition III.18(ii), we denote the existential topology on $\mathcal{C} \rtimes P$ induced by the existential doctrinal site $(P, (J_{\mathrm{Coh}})_{c \in \mathcal{C}})$ by J_{Coh}, i.e. the Grothendieck topology generated by covering families of the form

$$(d, x) \xrightarrow{g} (c, \exists_g x \vee \exists_h y) \xleftarrow{h} (e, y),$$

for $x \in P(d)$, $y \in P(d)$, and arrows $d \xrightarrow{g} c, e \xrightarrow{h} c \in \mathcal{C}$. We call the resultant doctrinal site (P, J_{Coh}) a *relative coherent doctrine* and denote the 1-full 2-subcategory of **DocSites** on relative coherent doctrines by **RelCohDoc**. We once again have that **CohDoc** is a 1-full 2-subcategory of **RelCohDoc**.

III.3.2 Syntactic categories

Recall that each cartesian category $\mathcal{C}$ yields a primary doctrine via the doctrine of subobjects $\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$. Taking the doctrine of subobjects of a cartesian category naturally defines a 2-functor

$$\mathrm{Sub}_{(-)} \colon \mathbf{Cart} \longrightarrow \mathbf{PrimDoc}.$$

(i) Each cartesian functor $F \colon \mathcal{C} \to \mathcal{D}$ restricts to a meet-semilattice homomorphism $a^F_c \colon \mathrm{Sub}_{\mathcal{C}}(c) \to \mathrm{Sub}_{\mathcal{D}}(F(c))$ natural in $c \in \mathcal{C}$. Hence, the pair (F, a^F) defines a morphism of primary doctrines $(F, a^F) \colon \mathrm{Sub}_{\mathcal{C}} \to \mathrm{Sub}_{\mathcal{D}}$.

(ii) Each natural transformation $\alpha \colon F \Rightarrow F'$ defines a 2-cell

$$\begin{tikzcd} \mathrm{Sub}_{\mathcal{C}} \arrow[r, bend left, "{(F,a^F)}"] \arrow[r, bend right, "{(F',a^{F'})}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Sub}_{\mathcal{D}} \end{tikzcd}$$

of **PrimDoc**. The required inequality $a^F(x) \leqslant \mathrm{Sub}_C(\alpha_c)(a^{F'}(x))$, for each $c \in C$ and each $x \in \mathrm{Sub}_C(c)$, follows by the universal property of the pullback

$$\begin{tikzcd}
F(x) \arrow[r, dashed] \arrow[rr, bend left, "\alpha_x"] \arrow[dr, tail] & \mathrm{Sub}_C(\alpha_c)(F'(x)) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & F'(x) \arrow[d, tail] \\
& F(c) \arrow[r, "\alpha_c"] & F'(c).
\end{tikzcd}$$

Moreover, the 2-functor $\mathrm{Sub}_{(-)}$ can easily be checked to be full and faithful on 1-cells.

From a doctrine $P\colon C^{\mathrm{op}} \to$ **PreOrd** with sufficient structure, it is possible to construct a *syntactic category* **Syn**(P). This construction is converse to taking the doctrine of subobjects in the sense that **Syn**$(\mathrm{Sub}_C) \simeq C$ for a certain subclass of cartesian categories, namely regular categories. It is only possible to construct the syntactic category of a doctrine P when P is rich enough to interpret *provably functional relations*. These are predicates that, according to the internal logic of a doctrine (see [55, §4.3] or below), encode the graph of a function between two other predicates. For that, we need at least regular logic. In this subsection we review material found in [98], [99] and [100] regarding the syntactic category construction. Our exposition is similar to the explanation found in [30, §3].

The internal language of a doctrine. As explicated in [55, §4.3], we can transliterate the structure of an existential doctrine into a more familiar logical language.

Let $P\colon C^{\mathrm{op}} \to$ **MSLat** be an existential doctrine.

(i) If $U \in P(c_1 \times \ldots \times c_n)$, we will write $U(x_1, \ldots, x_n)$.

(ii) For $U, V \in P(c)$, we will write $U(x) \wedge V(x)$ instead of $(U \wedge V)(x)$.

(iii) For an arrow $d \xrightarrow{f} c \in C$ and $U \in P(c)$, we will write $U(f(x))$ in place of $P(f)(U)$.

(iv) Given objects $c, d \in C$ and $W \in P(c \times d)$, we will write $\exists y : d\ W(x, y)$ in place of $\exists_{P(\mathrm{pr}_2)} W$, where pr_2 is the projection

$$c \times d \xrightarrow{\mathrm{pr}_2} d.$$

(v) Given $U \in P(c)$, we will write $U(x_1) \wedge x_1 = x_2$ in place of $\exists_{\Delta_c} U$, where Δ_c is the diagonal

$$c \xrightarrow{\Delta_c} c \times c.$$

(vi) More generally, for any arrow $d \xrightarrow{f} c$ and $V \in P(d)$, we will write

$$\exists y : d\ V(y) \wedge f(y) = x$$

for $\exists_{P(f)} V$.

(vii) Finally, given $U, V \in P(c)$ with $U \leqslant V$, we will write

$$U(x) \vdash_{x:c} V(x).$$

Inequalities in the fibres of a doctrine are thus lent a logical intuition when rewritten as sequents in this manner. For example, is it easier to intuit the validity of the equation

$$\exists_{\Delta_{c\times d}} \top_{c\times d} = P(\mathrm{pr}_{1,3})(\exists_{\Delta_c} \top_c) \wedge P(\mathrm{pr}_{2,4})(\exists_{\Delta_d} \top_d),$$

where $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,4}$ are the projections

$$\begin{tikzcd} & c \times d \times c \times d \arrow[dl, bend right, "\mathrm{pr}_{2,4}"'] \arrow[dr, bend left, "\mathrm{pr}_{1,3}"] & \\ c \times c & & d \times d, \end{tikzcd}$$

or the validity of the equivalence $(x, y) = (x', y') \dashv\vdash_{x,x':c;\, y,y':d} x = x' \wedge y = y'$? (See [78] for an entirely categorical proof of the former).

In fact, this transliteration can be formalised into a sequent calculus, as is done in [55, §4.3], in which the symbols used in the transcription can be manipulated as one would expect them to be, and the calculus has a complete and sound interpretation in existential doctrines – meaning that a sequent can be proven in the calculus if and only if the corresponding inequality is satisfied in every existential doctrine.

In this section, we will at times use the internal language of an existential doctrine to intuit results whose explicit demonstrations would be tangentially tedious to our exposition. Elementary proofs, without the use of the internal language of a doctrine, are provided in Appendix A.

Building a syntactic category. It seems superfluous to recall that, given two subsets $A \subseteq X$ and $B \subseteq Y$, the graph of a function $f : A \to B$ consists of a subset $f \subseteq X \times Y$ such that

$$\begin{aligned} (x, y) \in f &\implies x \in A, y \in B, \\ (x, y),\ (x, y') \in f &\implies y = y', \\ x \in A &\implies \exists\, y \in B\ (x, y) \in f. \end{aligned}$$

The conceit behind provably functional relations is to translate these implications into the internal language of a doctrine.

Definition III.32. Let $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ be an existential doctrine. The *syntactic category* $\mathbf{Syn}(P)$ of P is the category:

(i) whose objects are pairs (c, U) where c is an object of $\mathcal{C}$ and $U \in P(c)$,

(ii) and each arrow $(c, U) \to (d, V)$ is given by some $W \in P(c \times d)$ that defines a *provably functional relation*, i.e. the sequents

$$\begin{aligned} W(x, y) &\vdash_{x:c;\, y:d} U(x) \wedge V(y), \\ W(x, y) \wedge W(x, y') &\vdash_{x:c;\, y,y':d} y = y', \\ U(x) &\vdash_{x:c} \exists y : d\ W(x, y) \end{aligned}$$

are derivable in the internal language of P, or more concretely the inequalities

$$\begin{aligned} W &\leqslant P(\mathrm{pr}_1)(U) \wedge P(\mathrm{pr}_2)(V), \\ P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W) &\leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d} \top_d, \\ U &\leqslant \exists_{\mathrm{pr}_1} W \end{aligned}$$

are satisfied, where pr_1 and pr_2 are the projections

$$c \xleftarrow{\mathrm{pr}_1} c \times d \xrightarrow{\mathrm{pr}_2} d,$$

$\mathrm{pr}_{1,2}$, $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$ are the projections

$$\begin{tikzcd}
 & c \times d \times d \arrow[dl, bend right, "\mathrm{pr}_{1,3}"'] \arrow[d, "\mathrm{pr}_{1,2}"] \arrow[dr, bend left, "\mathrm{pr}_{2,3}"] & \\
c \times d & c \times d & d \times d,
\end{tikzcd}$$

and $\Delta_d \colon d \to d \times d$ is the diagonal.

The identity morphism on (c, U) is given by $\exists_{\Delta_c} U \in P(c \times c)$, while the composite of two arrows

$$(c, U) \xrightarrow{W} (d, V) \xrightarrow{W'} (e, V')$$

is given by the composite of W and W' as relations, i.e.

$$\exists_{\mathrm{pr}_{1,3}} \left(P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{2,3})(W') \right),$$

which is $\exists y : d \; W(x, y) \wedge W'(y, z)$ in the internal language.

Remark III.33. We have presented the syntactic category construction for an existential doctrine. This is analogous to the *category of maps* construction for an allegory (see [39, §2.132]). Indeed, for each existential doctrine P, the two constructions coincide in that $\mathbf{Syn}(P) \simeq \mathcal{MAP}(\mathbf{A}(P))$, where $\mathbf{A}(P)$ is the allegory of relations on P, i.e. the allegory whose objects are elements $U \in P(c \times d)$ (note that P must be a regular doctrine in order to express the relational composite of $U \in P(c \times d)$ and $V \in P(d \times e)$).

Example III.34. Let $\mathbb{T}$ be a theory in a fragment of first order logic that contains regular logic, i.e. the symbols $\{\wedge, \top, \exists\}$ (see [63, Defintion D1.1.3]). The syntactic category $\mathbf{Syn}(F^{\mathbb{T}})$ of the existential doctrine $F^{\mathbb{T}} \colon \mathbf{Con}_N \to \mathbf{MSLat}$, by definition, coincides with the usual syntactic category $\mathcal{C}_{\mathbb{T}}$ for $\mathbb{T}$ as described in [63, §D1.4].

The syntax-subobject adjunction. Taking the syntactic category of an existential doctrine yields a left inverse to the restriction of the 2-functor $\mathrm{Sub}_{(-)} \colon \mathbf{Cart} \to \mathbf{PrimDoc}$ to a suitable 2-subcategory – the 2-category of regular categories.

Definition III.35 (§A1.3 [63])**.** By **Reg**, we denote the 2-category:

(i) whose objects are *regular categories* – categories with finite limits and image factorisations that are stable under pullback (we will also require that a regular category is *well-powered* – i.e. each object has a small set of subobjects),

(ii) whose 1-cells are *regular functors* (also called *exact functors* in [6] and [28]) – finite limit preserving functors that also preserve regular epimorphisms,

(iii) and whose 2-cells are natural transformations between regular functors.

For each regular category C, the subobject doctrine $\mathrm{Sub}_C\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ is an existential doctrine (see [55, Theorem 4.4.4]). The left adjoint to $\mathrm{Sub}_C(f)$, for an arrow f of C, is given by the existence of images (see [63, Lemma A1.3.1]). Thus, the subobject 2-functor $\mathrm{Sub}_{(-)}\colon \mathbf{Cart} \to \mathbf{PrimDoc}$ from above restricts to a 2-functor

$$\mathrm{Sub}_{(-)}\colon \mathbf{Reg} \longrightarrow \mathbf{ExDoc}.$$

The syntactic category construction also induces a 2-functor $\mathbf{Syn}\colon \mathbf{ExDoc} \to \mathbf{Reg}$. For an existential doctrine P, the category $\mathbf{Syn}(P)$ is regular: it has finite limits and the image factorisation of an arrow $(c, U) \xrightarrow{W} (d, V) \in \mathbf{Syn}(P)$ is given by

$$(c, U) \overset{W}{\twoheadrightarrow} (d, \exists_{\mathrm{pr}_2} W) \rightarrowtail (d, V)$$

(see [98, §2.4 & 2.5]). A morphism of existential doctrines $(F, a)\colon P \to Q$ preserves the interpretation of regular logic, and therefore (F, a) induces a regular functor $\mathbf{Syn}(F, a)\colon \mathbf{Syn}(P) \to \mathbf{Syn}(Q)$.

The two functors form a pseudo-adjunction

$$\begin{tikzcd} \mathbf{ExDoc} \arrow[r, shift left=2, "\mathbf{Syn}"] \arrow[r, phantom, "\perp"] & \mathbf{Reg}. \arrow[l, shift left=2, "\mathrm{Sub}_{(-)}"] \end{tikzcd} \tag{III.ii}$$

By *pseudo-adjunction* we mean that, rather than there being a natural isomorphism of hom-categories, there is instead only a natural equivalence. Since $\mathrm{Sub}_{(-)}$ is full and faithful on 1-cells, the counit is a natural equivalence of categories $\mathbf{Syn}(\mathrm{Sub}_C) \simeq C$ for each $C \in \mathbf{Reg}$. The pseudo-adjunction (III.ii) can be deduced from the analogous pseudoadjunction found in [99, Proposition 1.3] and [100, Theorem 3.6].

III.3.3 Syntactic sites

We desire a version of the pseudo-adjunction (III.ii) that also incorporates Grothendieck topologies. To that end, we introduce the 2-category of *regular sites* and a particular 2-category of existential doctrinal sites which we denote by **ExDocSites**. We then extend the 2-functors $\mathbf{Syn}\colon \mathbf{ExDoc} \to \mathbf{Reg}$ and $\mathrm{Sub}_{(-)}\colon \mathbf{Reg} \to \mathbf{ExDoc}$ to these 2-categories by showing that for each existential doctrinal site $(P, (J_c)_{c \in C}) \in \mathbf{ExDocSites}$ (respectively, regular site (C, K)) there exists a natural choice of Grothendieck topology $J_{\mathbf{Syn}}$ on $\mathbf{Syn}(P)$ making $(\mathbf{Syn}(P), J_{\mathbf{Syn}})$ a regular site (resp., a natural choice of Grothendieck topology $K|_{\mathrm{Sub}_C(c)}$ on $\mathrm{Sub}_C(c)$, for each $c \in C$, making $(\mathrm{Sub}_C, (K|_{\mathrm{Sub}_C(c)})_{c \in C})$ an existential site). Finally, we demonstrate that these extended 2-functors are pseudo-adjoint.

Definition III.36. (i) Let **RegSites** be the 2-category whose objects are *regular sites*, which are sites (C, K) where C is a regular category and K is a Grothendieck topology on C such that the sieve generated by each regular epimorphism $c \twoheadrightarrow d$ is a K-cover. The 1-cells of **RegSites** are cover preserving regular functors and the 2-cells are all natural transformations between these.

(ii) By **ExDocSites** we denote the the 1-full 2-subcategory of **DocSites** on objects of the form $(P, J_{\rtimes})$ for an existential doctrinal site $(P, (J_c)_{c \in C})$ whose underlying doctrine P is also a primary doctrine (and therefore, by Proposition III.30, an existential doctrine).

Remark III.37. Let $(C, K) \in \mathbf{RegSites}$ be a regular site. As every morphism $d \xrightarrow{f} c \in C$ can be factored as a regular epimorphism followed by a monomorphism

$$d \twoheadrightarrow f(d) \rightarrowtail c,$$

the Grothendieck topology K on C is entirely determined by which families of subobjects $\{ e_i \rightarrowtail c \mid i \in I \}$ are K-covering for each $c \in C$.

For a regular site (C, K), by endowing each subobject lattice $\mathrm{Sub}_C(c)$ with the Grothendieck topology $K|_{\mathrm{Sub}_C(c)}$, we obtain an existential site $(C, (K|_{\mathrm{Sub}_C(c)})_{c \in C})$. We note that the left adjoint

$$\exists_f \colon \mathrm{Sub}_C(c) \longrightarrow \mathrm{Sub}_C(d),$$

for each arrow $c \xrightarrow{f} d \in C$, preserves covers. If a family of arrows $\{ a_i \rightarrowtail b \,|\, i \in I \}$ in $\mathrm{Sub}_C(c)$ is K-covering then, using the diagram

$$\begin{tikzcd}
a_i \arrow[r, tail] \arrow[dr, tail] \arrow[rr, bend left, two heads] & b \arrow[d, tail] \arrow[rr, bend left, two heads] & f(a_i) \arrow[r, tail] \arrow[dr, tail] & f(b) \arrow[d, tail] \\
& c \arrow[rr, "f"] & & d,
\end{tikzcd}$$

the fact that $a_i \twoheadrightarrow f(a_i)$ and $b \twoheadrightarrow f(b)$ are both K-covers, and the fact that K satisfies the transitivity condition, we observe that $\{ f(a_i) \rightarrowtail f(b) \,\big|\, i \in I \}$ is a K-covering family too.

We denote the resulting topology on $C \rtimes \mathrm{Sub}_C$ by K_{Sub}. It is easily checked that, given regular sites (C, K) and $(\mathcal{D}, K')$ and a regular functor $F \colon C \to \mathcal{D}$, F sends K-covers to K'-covers if and only if the induced morphism of doctrines $(F, a^F) \colon \mathrm{Sub}_C \to \mathrm{Sub}_{\mathcal{D}}$ sends K_{Sub}-covers to K'_{Sub}-covers. Thus, we obtain a 2-functor

$$\mathrm{Sub}_{(-)} \colon \mathbf{RegSites} \longrightarrow \mathbf{ExDocSites}$$

which, moreover, is full and faithful on 1-cells since $\mathrm{Sub}_{(-)} \colon \mathbf{Reg} \to \mathbf{ExDoc}$ is full and faithful on 1-cells.

Conversely, for each existential site $(P, (J_c)_{c \in C}) \in \mathbf{ExDocSites}$, we can endow the syntactic site $\mathbf{Syn}(P)$ with the Grothendieck topology $J_{\mathbf{Syn}}$ where a family of arrows

$$\left\{ (c_i, U_i) \xrightarrow{W_i} (d, V) \,\middle|\, i \in I \right\}$$

is $J_{\mathbf{Syn}}$-covering if and only if the family

$$\{ \exists_{\mathrm{pr}_2} W_i \leqslant V \mid i \in I \}$$

is J_d-covering. Recall that the image factorisation of an arrow $(c, U) \xrightarrow{W} (d, V) \in \mathbf{Syn}(P)$ is given by

$$(c, U) \overset{W}{\twoheadrightarrow} (d, \exists_{\mathrm{pr}_2} W) \rightarrowtail (d, V),$$

whose left factor $(c, U) \xrightarrow{W} (d, \exists_{\mathrm{pr}_2} W)$ is trivially a $J_{\mathbf{Syn}}$-cover. Thus, regular epimorphisms are J_{Syn}-covers and so $(\mathbf{Syn}(P), J_{\mathbf{Syn}})$ is a regular site. It is equally trivially observed that if $(F, a)\colon (P, J_{\rtimes}) \to (Q, J'_{\rtimes})$ is a morphism of **ExDocSites** then the induced functor $\mathbf{Syn}(F, a)\colon \mathbf{Syn}(P) \to \mathbf{Syn}(Q)$ sends $J_{\mathbf{Syn}}$-covers to $J'_{\mathbf{Syn}}$-covers, and hence there is a functor

$$\mathbf{Syn}\colon \mathbf{ExDocSites} \longrightarrow \mathbf{RegSites}.$$

Proposition III.38. *The pseudo-adjunction (III.ii) extends to give a second pseudo-adjunction and a morphism of adjunctions*

$$\begin{tikzcd}
\mathbf{ExDoc} \arrow[d, shift right=2, "\mathbf{Syn}"'] \arrow[d, phantom, "\dashv"] & \mathbf{ExDocSites} \arrow[l, "U"'] \arrow[d, shift right=2, "\mathbf{Syn}"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Reg} \arrow[u, shift right=2, "\mathrm{Sub}_{(-)}"'] & \mathbf{RegSites}, \arrow[l, "U'"'] \arrow[u, shift right=2, "\mathrm{Sub}_{(-)}"']
\end{tikzcd}$$

i.e. $\mathbf{Syn} \circ U = U' \circ \mathbf{Syn}$ *and* $\mathrm{Sub}_{(-)} \circ U' = U \circ \mathrm{Sub}_{(-)}$, *where U and U' are the forgetful 2-functors.*

Proof. For each pair of a regular site $(C, K) \in \mathbf{RegSites}$ and an existential doctrinal site $(P, (J_c)_{c \in C})$, the necessary equivalence

$$\mathbf{RegSites}((\mathbf{Syn}(P), J_{\mathbf{Syn}}), (C, K)) \simeq \mathbf{ExDocSites}((P, J_{\rtimes}), (\mathrm{Sub}_C, K_{\mathrm{Sub}})) \tag{III.iii}$$

is easily obtained by restricting the equivalence $\mathbf{Reg}(\mathbf{Syn}(P), C) \simeq \mathbf{ExDoc}(P, \mathrm{Sub}_C)$ to those regular functors $\mathbf{Syn}(P) \to C$ (respectively, morphisms of existential doctrines $P \to \mathrm{Sub}_C$) which are cover preserving (resp., for which the induced functors $\mathcal{D} \rtimes P \to C \rtimes \mathrm{Sub}_C$ are cover preserving). □

III.3.4 Syntactic sites versus doctrinal sites

For each existential doctrinal site $(P, (J_c)_{c \in C}) \in \mathbf{ExDocSites}$, there are now two choices, $\mathbf{Sh}(C \rtimes P, J_{\rtimes})$ and $\mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}})$, for the topoi we can associate with the doctrinal site. In particular, if $\mathbb{T}$ is a theory in a fragment of logic that contains regular logic, then there are two choices of site for the classifying topos of $\mathbb{T}$ – one built from the doctrine associated to $\mathbb{T}$, and the other built from the syntactic category.

However, already by the natural equivalence (III.iii), we can deduce a natural equivalence

$$\begin{aligned}
\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}})) &\simeq \mathbf{RegSites}((\mathbf{Syn}(P), J_{\mathbf{Syn}}), (\mathcal{E}, J_{\mathrm{can}})), \\
&\simeq \mathbf{ExDocSites}((P, J_{\rtimes}), (\mathrm{Sub}_{\mathcal{E}}, K_{\mathrm{Sub}_{\mathcal{E}}})), \\
&\simeq \mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(C \rtimes P, J_{\rtimes}))
\end{aligned}$$

for each topos $\mathcal{E}$. Hence, there is an equivalence of topoi

$$\mathbf{Sh}(C \rtimes P, J_{\rtimes}) \simeq \mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}}), \tag{III.iv}$$

and so it is equivalent, at the level of topos theory, to represent a theory using a doctrinal site or a syntactic site.

Despite this, it is instructive to see where this equivalence comes from. For each $(P, (J_c)_{c\in C}) \in \mathbf{ExDocSites}$, we will construct a functor $\zeta^P : C \rtimes P \to \mathbf{Syn}(P)$ and then demonstrate that

$$\zeta^P : (C \rtimes P, J_{\rtimes}) \to (\mathbf{Syn}(P), J_{\mathbf{Syn}})$$

is a dense morphism of sites, from which we deduce the equivalence (III.iv) by Lemma I.8.

Since $C \rtimes P$ and $\mathbf{Syn}(P)$ share the same objects, it is obvious how we would wish ζ^P to act on objects. Our first task therefore is to conjure a provably functional relation from an arrow $(c, U) \xrightarrow{f} (d, V)$ of $C \rtimes P$.

Lemma III.39. *Let $P : C^{\text{op}} \to \mathbf{MSLat}$ be an existential doctrine. For an arrow $(c, U) \xrightarrow{f} (d, V)$ of $C \rtimes P$, i.e. whenever $U \leqslant P(f)(V)$, the proposition*

$$\exists_{\text{id}_c \times f} U \in P(c \times d)$$

is a provably functional relation $(c, U) \to (d, V) \in \mathbf{Syn}(P)$.

In the internal language of P, $\exists_{\text{id}_c \times f} U$ is written as $U(x) \wedge f(x) = y$, while the inequality $U \leqslant P(f)(V)$ becomes $U(x) \vdash_{x:c} V(f(x))$. Written this way, the sequents

$$\begin{aligned} U(x) \wedge f(x) = y &\vdash_{x:c;\, y:d} V(y), \\ U(x) \wedge f(x) = y \wedge U(x) \wedge f(x) = y' &\vdash_{x:c;\, y,y':d} y = y', \\ U(x) \wedge f(x) = y &\vdash_{x:c;\, y:d} \exists y' : d\; U(x) \wedge f(x) = y' \end{aligned}$$

required for $\exists_{\text{id}_c \times f} U$ to be provably functional are evidently satisfied.

The assignments

$$\begin{aligned} (c, U) &\mapsto (c, U), \\ (c, U) \xrightarrow{f} (d, V) &\mapsto (c, U) \xrightarrow{\exists_{\text{id}_c \times f} U} (d, V) \end{aligned}$$

define a functor $\zeta^P : C \rtimes P \to \mathbf{Syn}(P)$. By definition, ζ^P preserves identities. That ζ^P preserves the composite

$$(c, U) \xrightarrow{\ f\ } (d, V) \xrightarrow{\ g\ } (e, W)$$

is expressed by the equivalence

$$\exists y : d\; (U(x) \wedge f(x) = y \wedge V(y) \wedge g(y) = z) \dashv\vdash_{x:c, z:e} U(x) \wedge g \circ f(x) = z$$

in the internal language of the doctrine P. Elementary demonstrations of Lemma III.39 and the functoriality of ζ^P are given in Appendix A.

Intuitively, the functor $\zeta^P : C \rtimes P \to \mathbf{Syn}(P)$ is 'adjoining those arrows that ought to exist' (i.e. those for which a provably functional relation exists) and 'identifying those arrows that ought to be the same' (i.e. those for which the internal language of P proves an identity of arrows).

Proposition III.40. *Let $(P, (J_c)_{c\in C})$ be an existential doctrinal site. The functor ζ^P defines a dense morphism of sites*

$$\zeta^P : (C \rtimes P, J_{\rtimes}) \longrightarrow (\mathbf{Syn}(P), J_{\mathbf{Syn}}),$$

and hence there is an equivalence of topoi $\mathbf{Sh}(C \rtimes P, J_{\rtimes}) \simeq \mathbf{Sh}(\mathbf{Syn}(P), J_{\mathbf{Syn}})$.

Proof. We check the four conditions of Definition I.6 one by one.

(i) The first condition, Definition I.6(i), is immediate once we recall that a family of morphisms

$$\left\{ (c_i, U_i) \xrightarrow{f_i} (d, V) \,\middle|\, i \in I \right\} \text{ in } \mathcal{C} \rtimes P$$

is $J_{\rtimes}$-covering if and only if

$$\left\{ \exists_{f_i} U_i = \exists_{\mathrm{pr}_2} \exists_{\mathrm{id}_{c_i} \times f_i} U_i \to V \,\middle|\, i \in I \right\} \text{ in } P(d)$$

is J_d-covering, if and only if the family of morphisms

$$\left\{ (c_i, U_i) \xrightarrow{\exists_{\mathrm{id}_{c_i} \times f_i} U_i} (d, V) \,\middle|\, i \in I \right\} \text{ in } \mathbf{Syn}(P)$$

is $J_{\mathbf{Syn}}$-covering.

(ii) Condition Definition I.6(ii) follows since the functor ζ^P is surjective on objects.

(iii) Let $(c, U) \xrightarrow{W} (d, V)$ be a provably functional relation, i.e. an arrow of $\mathbf{Syn}(P)$. As $W \leqslant P(\mathrm{pr}_1)(U)$, there is an arrow $(c \times d, W) \xrightarrow{\mathrm{pr}_1} (c, U)$ of $\mathcal{C} \rtimes P$. Consider the diagram

$$(c \times d, W) \xrightarrow{\zeta^P(\mathrm{pr}_1) = \exists_{\mathrm{id}_{c \times d} \times \mathrm{pr}_1} W} (c, U) \xrightarrow{\quad W \quad} (d, V) \tag{III.v}$$

in $\mathbf{Syn}(P)$. To satisfy condition Definition I.6(iii), it suffices to show that the arrow $(c \times d, W) \xrightarrow{\zeta^P(\mathrm{pr}_1)} (c, U)$ is $J_{\mathbf{Syn}}$-covering and the composite of (III.v) is in the image of ζ^P. The former follows from the inequality

$$U \leqslant \exists_{\mathrm{pr}_1} W = \exists_{\mathrm{pr}_2} \exists_{\mathrm{id}_{c \times d} \times \mathrm{pr}_1} W$$

while the latter is expressed by the equivalences

$$\begin{aligned} \exists x' : c \; W(x, y) \wedge x = x' \wedge W(x', y') &\dashv\vdash_{x:c;\, y, y':d} W(x, y) \wedge W(x, y'), \\ &\dashv\vdash_{x:c;\, y, y':d} W(x, y) \wedge y = y' \end{aligned}$$

in the internal language of P. An elementary proof is provided in Lemma A.3 in Appendix A.

(iv) Let

$$\begin{tikzcd} (c, U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (d, V) \end{tikzcd}$$

be a pair of parallel arrows of $\mathcal{C} \rtimes P$ that are identified in the image of ζ^P, i.e. $\exists_{\mathrm{id}_c \times f} U = \exists_{\mathrm{id}_c \times g} U$. To satisfy condition Definition I.6(iv), we aim to find a $J_{\rtimes}$-cover S of (c, U) such that, for all $h \in S$, $f \circ h = g \circ h$. Let

$$\begin{tikzcd} e \arrow[r, "h"] & c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & d \end{tikzcd}$$

be an equalizer diagram in $\mathcal{C}$, and hence the square

$$\begin{tikzcd} e \arrow[r, "h"] \arrow[d, "h"'] & c \arrow[d, "\mathrm{id}_c \times f"] \\ c \arrow[r, "\mathrm{id}_c \times g"] & c \times d \end{tikzcd} \tag{III.vi}$$

is a pullback. The fork

$$(c, P(h)(U)) \xrightarrow{h} (c, U) \underset{g}{\overset{f}{\rightrightarrows}} (d, V)$$

in $\mathcal{C} \rtimes P$ commutes, and the arrow $(c, P(h)(U)) \xrightarrow{h} (c, U)$ is $J_{\rtimes}$-covering since

$$\begin{aligned} \exists_{\mathrm{id}_c \times f} U = \exists_{\mathrm{id}_c \times g} U &\implies U \leqslant P(\mathrm{id}_c \times f)\exists_{\mathrm{id}_c \times g} U, \\ &\implies U \leqslant \exists_h P(h)(U), \end{aligned}$$

where the last implication is an application of the Beck-Chevalley condition for the square (III.vi).

□

The choice of functor ζ^P is suitably natural. Recall that **MorphSites** denotes the bicategory of sites, morphisms of sites and natural transformations between these. The two sites that can be assigned to an existential doctrinal site $(P, (J_c)_{c \in \mathcal{C}}) \in$ **ExDocSites**,

$$(P, (J_c)_{c \in \mathcal{C}}) \mapsto (\mathcal{C} \rtimes P, J_{\rtimes}) \text{ and } (P, (J_c)_{c \in \mathcal{C}}) \mapsto (\mathbf{Syn}(P), J_{\mathbf{Syn}}),$$

yield a pair of bifunctors

$$\mathbf{ExDocSites} \underset{\mathbf{Syn}}{\overset{\rtimes}{\rightrightarrows}} \mathbf{MorphSites}.$$

It is easily checked that the morphisms of sites $\zeta^P : (\mathcal{C} \rtimes P, J_{\rtimes}) \to (\mathbf{Syn}(P), J_{\mathbf{Syn}})$ is the component at the existential doctrinal site $(P, (J_c)_{c \in \mathcal{C}})$ of a natural transformation

$$\begin{tikzcd} \mathbf{ExDocSites} \arrow[r, bend left, "\rtimes"] \arrow[r, bend right, "\mathbf{Syn}"'] \arrow[r, phantom, "\Downarrow \zeta"] & \mathbf{MorphSites} \end{tikzcd}$$

Since ζ^P is a dense morphism of sites for each $(P, (J_c)_{c \in \mathcal{C}})$, the composite 2-cell $\mathbf{Sh} * \zeta$,

$$\begin{tikzcd} \mathbf{ExDocSites} \arrow[r, bend left, "\rtimes"] \arrow[r, bend right, "\mathbf{Syn}"'] \arrow[r, phantom, "\Downarrow \zeta"] & \mathbf{MorphSites} \arrow[r, "\mathbf{Sh}"] & \mathbf{Topos}, \end{tikzcd}$$

is an isomorphism.

Remark III.41. Let $\mathbb{T}$ be a geometric theory over a signature Σ with N sorts. (See Section III.4 for more on geometric theories; also this remark, with suitable modifications, will apply to any theory in a fragment of first-order logic that contains regular logic.) A textbook account of classifying topos theory, as can be found in [63, §D1.4], [79, §X] or [22, §1.4], presents the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ with the *syntactic site* $(C_{\mathbb{T}}, J_{\mathbb{T}})$.

(i) The syntactic category $C_{\mathbb{T}}$ of $\mathbb{T}$ is the category

a) whose objects are the $\mathbb{T}$-provable equivalence classes of formulae $\{\, \vec{x} : \varphi \,\}$ over Σ,

b) and whose arrows $\{\, \vec{x} : \varphi \,\} \xrightarrow{[\theta]} \{\, \vec{y} : \psi \,\}$ are $\mathbb{T}$-provable equivalence classes of $\mathbb{T}$-provably functional formulae, that is formulae θ in the context $\vec{x}, \vec{y}$ such that $\mathbb{T}$ proves the sequents

$$\theta \vdash_{\vec{x},\vec{y}} \varphi \wedge \psi, \quad \varphi \vdash_{\vec{x}} \exists \vec{y}\, \theta, \quad \theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}.$$

(ii) In the syntactic topology $J_{\mathbb{T}}$ on $C_{\mathbb{T}}$, a family of arrows

$$\left\{ \{\, \vec{x_i} : \varphi_i \,\} \xrightarrow{[\theta_i]} \{\, \vec{y} : \psi \,\} \,\middle|\, i \in I \right\}$$

is $J_{\mathbb{T}}$-covering if and only if $\mathbb{T}$ proves the sequent

$$\psi \vdash_{\vec{y}} \bigvee_{i \in I} \exists \vec{x_i}\, \theta_i.$$

We immediately recognise the category $C_{\mathbb{T}}$ as the syntactic category construction $\mathbf{Syn}(F^{\mathbb{T}})$ for the doctrine $F^{\mathbb{T}} \colon \mathbf{Con}_N \to \mathbf{Frm}$ associated with the theory $\mathbb{T}$. Similarly, the syntactic topology $J_{\mathbb{T}}$ is precisely the topology $J_{\mathbf{Syn}}$ obtained from the existential doctrinal site $(F^{\mathbb{T}}, (J_{\vec{x}})_{\vec{x} \in \mathbf{Con}_N})$, where each fibre $F^{\mathbb{T}}(\vec{x})$ has been endowed with the canonical topology on the frame. (The induced topology $J_{\rtimes}$ on $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$ is precisely the topology $K_{F^{\mathbb{T}}}$ from Definition II.11.)

Thus, by Proposition III.40, we conclude that both $(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, J_{\rtimes})$ and $(C_{\mathbb{T}}, J_{\mathbb{T}})$ are both sites of definition for the classifying topos $\mathcal{E}_{\mathbb{T}}$, as visualised in the 'bridge' diagram

$$\begin{tikzcd}
& \begin{array}{c}\mathcal{E}_{\mathbb{T}}\\ \textit{the classifying}\\ \textit{topos of } \mathbb{T},\end{array} \arrow[dl, leftrightarrow, dashed, bend right] \arrow[dr, leftrightarrow, dashed, bend left] & \\
\begin{array}{c} F^{\mathbb{T}}\colon \mathbf{Con}_N \to \mathbf{Frm}\\ \textit{the geometric doctrine,}\end{array} & & \begin{array}{c}(C_{\mathbb{T}}, J_{\mathbb{T}})\\ \textit{the syntactic site.}\end{array}
\end{tikzcd}$$

The site $(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}})$ was dubbed the *alternative syntactic site* of the theory $\mathbb{T}$ in [125].

Why would one choose one site of definition for $\mathcal{E}_{\mathbb{T}}$ over another? As we will make use of in Chapter VII, the site $(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}})$ can be more amenable than the standard syntactic site for some calculations. Notably, every arrow $(\vec{x}, \varphi) \xrightarrow{\sigma} (\vec{y}, \psi)$ is a restriction of the arrow

$$\begin{tikzcd} (\vec{x}, \varphi) \arrow[r, "\sigma"] \arrow[d, tail] & (\vec{y}, \psi) \arrow[d, tail] \\ (\vec{x}, \top) \arrow[r, "\sigma"] & (\vec{y}, \top), \end{tikzcd}$$

and moreover, since $(\vec{z}, \top)$ is the product $\prod_{z_i \in \vec{z}} (z_i, \top)$ in $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$, the arrow $(\vec{x}, \top) \xrightarrow{\sigma} (\vec{y}, \top)$, labelled by a relabelling $\sigma \colon \vec{y} \to \vec{x}$, is induced universally as in the diagram

$$\begin{tikzcd} (\vec{x}, \top) \arrow[r, dashed, "\sigma"] \arrow[d, "\mathrm{pr}_{\sigma(y_i)}"] & \prod\limits_{y_i \in \vec{y}} (y_i, \top) = (\vec{y}, \top) \arrow[d, "\mathrm{pr}_{y_i}"] \\ (\sigma(y_i), \top) \arrow[r, "\mathrm{id}_{\sigma(y_i)} = \mathrm{id}_{y_i}"] & (y_i, \top). \end{tikzcd}$$

Conversely, there are desirable properties of the syntactic site $(C_{\mathbb{T}}, J_{\mathbb{T}})$ that are not shared by the alternative site $(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}})$. For example, the topology $J_{\mathbb{T}}$ is subcanonical (see [79, Lemma X.4.5]) while the topology $K_{F^{\mathbb{T}}}$ on $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$ is not (see Remark II.15(ii)).

III.4 Geometric theories as internal locales

Finally, we focus exclusively on doctrines that interpret geometric logic. Let $\mathbb{T}$ be a theory of geometric logic, i.e. $\mathbb{T}$ is a theory in that fragment of first-order infinitary logic whose permissible symbols are $\{\wedge, \top, \bot, \bigvee, \exists, =\}$, over a signature Σ with N sorts. The doctrine $F^{\mathbb{T}} \colon \mathbf{Con}_N \to \mathbf{DLat}$ associated to the theory is an internal locale of the presheaf topos $\mathbf{Sets}^{\mathbf{Con}_N}$.

(i) The doctrine $F^{\mathbb{T}}$ takes values in the category **Frm**.

(ii) For each arrow $\vec{x} \xrightarrow{\sigma} \vec{y} \in \mathbf{Con}_N$, $F^{\mathbb{T}}(\sigma)$ has a left adjoint $\exists_{F^{\mathbb{T}}(\sigma)}$ – the left adjoint $\exists_{F^{\mathbb{T}}(\sigma)}$ acts on a formula $\varphi \in F^{\mathbb{T}}(\vec{y})$ by

$$\exists_{F^{\mathbb{T}}(\sigma)}(\varphi) = \left[\exists \vec{y}\, \varphi \wedge \bigwedge_{y_i \in \vec{y}} y_i = \sigma(y_i) \right].$$

(iii) Moreover, these left adjoints satisfy both the Frobenius and Beck-Chevalley conditions.

Thus, by the classification of internal locales of $\mathbf{Sets}^{\mathbf{Con}_N}$ given in [68, Proposition VI.2.2] (see also Theorem II.10), we obtain the following proposition, as observed in the single-sorted case in [63, Theorem D3.2.5] (the observation could also be obtained using the theory of *localic expansions* of [22, §7.1], see also Section III.4.1).

Proposition III.42 (Theorem D3.2.5 [63]). *An internal locale $\mathbb{L}$ of $\mathbf{Sets}^{\mathbf{Con}_N}$ corresponds, up to equivalence, to an N-sorted geometric theory $\mathbb{T}$ (the exact notion of equivalence for theories is provided by Theorem III.14).*

Geometric doctrines. We are therefore inclined to define a geometric doctrines as follows:

Definitions III.43. (i) A *geometric doctrine* is a doctrine that factors as

$$\mathbb{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}} \subseteq \mathbf{PreOrd}$$

and moreover satisfies one of the equivalent conditions:

a) $\mathbb{L}$ satisfies the relative Beck-Chevalley condition;
b) $\mathbb{L}$ defines an internal locale of the presheaf topos $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

(ii) We define the 2-category **GeomDoc** of geometric doctrines and their morphisms as the 1-full 2-subcategory $\mathbf{GeomDoc} \subseteq \mathbf{DocSites}$ on objects of the form $(\mathbb{L}, K_{\mathbb{L}})$, where the Grothendieck topology $K_{\mathbb{L}}$ on $\mathcal{C} \rtimes \mathbb{L}$ is the same topology as in Definition II.11, i.e. the topology where a sieve S in $\mathcal{C} \rtimes \mathbb{L}$ is $K_{\mathbb{L}}$-covering if and only if S contains a small family

$$\left\{ (c_i, U_i) \xrightarrow{f_i} (d, V) \,\middle|\, i \in I \right\}$$

in $\mathcal{C} \rtimes \mathbb{L}$ such that

$$V = \bigvee_{i \in I} \exists_{f_i} U_i.$$

Just as in Proposition III.18, we can identify a morphism of geometric doctrines $(F, a)\colon \mathbb{L} \to \mathbb{L}'$ as a pair consisting of a flat functor $F\colon \mathcal{C} \to \mathcal{D}$ between the base categories and a natural transformation $a\colon \mathbb{L} \Rightarrow \mathbb{L}' \circ F^{\mathrm{op}}$ where, for each $d \xrightarrow{g} c \in \mathcal{C}$, the square

$$\begin{tikzcd} \mathbb{L}_d \arrow[r, "\exists_{\mathbb{L}(g)}"] \arrow[d, "a_d"'] & \mathbb{L}_c \arrow[d, "a_c"] \\ \mathbb{L}'_{F(d)} \arrow[r, "\exists_{\mathbb{L}'(F(g))}"] & \mathbb{L}'_{F(c)} \end{tikzcd}$$

commutes. Thus, if $\mathbb{L}, \mathbb{L}'\colon \mathcal{C}^{\mathrm{op}} \rightrightarrows \mathbf{Frm}_{\mathrm{open}}$ are two internal locales fibred over the same category, then

$$\mathbf{GeomDoc}(\mathbb{L}, \mathbb{L}') = \mathbf{Loc}\left(\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}\right)(\mathbb{L}', \mathbb{L}).$$

Indeed, for any topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$, there is a 1-full 2-embedding

$$\mathbf{Loc}(\mathcal{E})^{\mathrm{op}} \hookrightarrow \mathbf{GeomDoc}.$$

Note also that we have not restricted ourselves to geometric doctrines that are fibred over cartesian categories. We will instead use $\mathbf{GeomDoc}_{\mathrm{cart}}$ to refer to the 1-full 2-subcategory of geometric doctrines that are fibred over cartesian categories.

Applications to geometric logic. Having identified geometric theories with internal locales, the remainder of this section is dedicated to applying our results on internal locales from Chapter II to deduce corresponding results on geometric theories. The results we prove were previously known in the literature via other methods. But, in the author's estimation, the perspective of internal locale theory yields the most elegant demonstrations of these facts.

III.4.1 Localic expansions

Our applications to geometric logic concern *expansions* of theories. Expansions of theories are ubiquitous in mathematics since, as soon as one notion is defined, it is natural to consider the same objects equipped with extra structure (formalised by *localic expansions*) or special cases (formalised by *quotient theories*). Over the next few results, we will observe that the structure of localic expansions is closely tied to theory of internal locales developed in Chapter II.

Definition III.44 (Definition 7.1.1 [22]). Let $\mathbb{T}$ be a geometric theory over a signature Σ. A *localic expansion* of $\mathbb{T}$ consists of an expanded signature $\Sigma' \supseteq \Sigma$ and a geometric theory $\mathbb{T}'$ over Σ' such that:

(i) the expanded signature Σ' adds no new sorts to the signature Σ,

(ii) the theory $\mathbb{T}'$ proves every every axiom $\varphi \vdash_{\vec{x}} \psi$ of the theory $\mathbb{T}$.

Corollary III.45 (Theorem 7.1.3 [22]). *For each localic expansion $\mathbb{T}'$ of $\mathbb{T}$, there is an induced localic geometric morphism*

$$e_{\mathbb{T}}^{\mathbb{T}'} : \mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}}.$$

Proof. For each context $\vec{x}$, we denote by $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$ the map

$$\begin{aligned} \mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}} : F^{\mathbb{T}}(\vec{x}) &\to F^{\mathbb{T}'}(\vec{x}), \\ [\varphi]_{\mathbb{T}} &\mapsto [\varphi]_{\mathbb{T}'}, \end{aligned}$$

where we have used the notation $[\varphi]_{\mathbb{T}}$ and $[\varphi]_{\mathbb{T}'}$ to differentiate between the class of formulae that are provably equivalent to φ according to the theory $\mathbb{T}$ and according to the theory $\mathbb{T}'$.

Since $\mathbb{T}'$ proves every axiom of $\mathbb{T}$, if $[\varphi]_{\mathbb{T}} \leqslant [\psi]_{\mathbb{T}}$, then $[\varphi]_{\mathbb{T}'} \leqslant [\psi]_{\mathbb{T}'}$, and so the map $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$ is monotone. Moreover, since

$$[\varphi \wedge \psi]_{\mathbb{T}} = [\varphi]_{\mathbb{T}} \wedge [\psi]_{\mathbb{T}} \text{ and } \left[\bigvee_{i\in I} \varphi_i\right]_{\mathbb{T}} = \bigvee_{i\in I} [\varphi]_{\mathbb{T}},$$

and similarly for $\mathbb{T}'$, the map $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$ is clearly also a frame homomorphism. Additionally, it is easily observed that $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$ is natural with respect to the maps $F^{\mathbb{T}}(\sigma)$ and $\exists_{F^{\mathbb{T}}(\sigma)}$ since $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$ also preserves substitution and the interpretation of the logical symbols $\{=, \exists\}$.

Therefore, the maps $\mathrm{e}^{\mathbb{T}'-1}_{\mathbb{T}\ \vec{x}}$, for each context $\vec{x} \in \mathbf{Con}_N$, are the components of an internal locale morphism $\mathrm{e}^{\mathbb{T}'}_{\mathbb{T}} : F^{\mathbb{T}'} \to F^{\mathbb{T}}$. Thus, by applying Proposition II.23, there is a localic geometric morphism

$$\mathbf{Sh}\left(\mathrm{e}^{\mathbb{T}'}_{\mathbb{T}}\right) : \mathbf{Sh}\left(F^{\mathbb{T}'}\right) \simeq \mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}\left(F^{\mathbb{T}}\right)$$

as desired, which we label by $e^{\mathbb{T}'}_{\mathbb{T}}$. □

Example III.46. Every theory $\mathbb{T}$ over a signature Σ with N sorts is a localic expansion of the N-sorted empty theory $N \cdot \mathbb{O}$. The induced localic geometric morphism

$$e^{\mathbb{T}}_{N\cdot\mathbb{O}} \colon \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{N\cdot\mathbb{O}} \simeq \mathbf{Sets}^{\mathbf{Con}_N}$$

is precisely the localic geometric morphism $C_{\pi_{F^{\mathbb{T}}}} : \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(F^{\mathbb{T}}) \to \mathbf{Sets}^{\mathbf{Con}_N}$ associated with the internal locale $F^{\mathbb{T}}$ of $\mathbf{Sets}^{\mathbf{Con}_N}$.

Conservative expansions. A localic expansion $\mathbb{T}'$ of a geometric theory $\mathbb{T}$ is said to be *conservative* if whenever $\mathbb{T}'$ proves a sequent $\varphi \vdash_{\vec{x}} \psi$ over the non-expanded signature Σ, then $\mathbb{T}$ also proves the sequent. Thus, an expansion $\mathbb{T}'$ is conservative if it proves no new theorems over the language of the original theory. It is easily observed that the expansion $\mathbb{T}'$ of $\mathbb{T}$ is conservative if and only if, for each context $\vec{x}$ over Σ, the frame homomorphism

$$\mathrm{e}^{\mathbb{T}'\,-1}_{\mathbb{T}\;\vec{x}} : F^{\mathbb{T}}(\vec{x}) \to F^{\mathbb{T}'}(\vec{x})$$

is injective. Thus, the following corollary is obtained by Proposition II.28;

Corollary III.47. *The induced localic geometric morphism $e^{\mathbb{T}'}_{\mathbb{T}} : \mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$ of a localic expansion $\mathbb{T}'$ of $\mathbb{T}$ is surjective if and only if $\mathbb{T}'$ is a conservative expansion.*

III.4.2 Quotient theories

A *quotient theory* is a particular kind of localic expansion where the signature does not change, i.e. a quotient theory is obtained by adding further axioms to the original theory. There is an equivalence between quotient theories and subtopoi, as found in [22, Theorem 3.2.5]. We give an elementary proof of this fact using internal sublocales.

Definition III.48 (Definition 3.2.2-4 [22])**.** Let $\mathbb{T}$ be a geometric theory.

(i) A *quotient theory* of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over the same signature Σ as $\mathbb{T}$ and which contains the axioms of $\mathbb{T}$.

(ii) Two quotient theories $\mathbb{T}', \mathbb{T}''$ of $\mathbb{T}$ are said to be *syntactically equivalent*, written as $\mathbb{T}' \equiv_s \mathbb{T}''$, if the axioms of $\mathbb{T}'$ are provable by the theory $\mathbb{T}''$ and vice-versa.

Corollary III.49 (Theorem 3.2.5 [22])**.** *Let $\mathbb{T}$ be a geometric theory. There is a bijective correspondence between the $\equiv_s$-equivalence classes of quotient theories of $\mathbb{T}$ and the subtopoi of $\mathcal{E}_{\mathbb{T}}$.*

Proof. A quotient theory $\mathbb{T}'$ of $\mathbb{T}$ is a particular case of a localic expansion of $\mathbb{T}$, and so, by Corollary III.45, there is an internal locale morphism

$$\mathrm{e}^{\mathbb{T}'}_{\mathbb{T}} : F^{\mathbb{T}'} \longrightarrow F^{\mathbb{T}}.$$

Moreover, since $\mathbb{T}'$ is a theory over the same signature as $\mathbb{T}$, for each context $\vec{x}$, the frame homomorphism $\mathrm{e}^{\mathbb{T}'\,-1}_{\mathbb{T}\;\vec{x}} : F^{\mathbb{T}}(\vec{x}) \to F^{\mathbb{T}'}(\vec{x})$ is evidently surjective. Thus, the internal locale morphism $\mathrm{e}^{\mathbb{T}'}_{\mathbb{T}}$ is an internal sublocale embedding $\mathrm{e}^{\mathbb{T}'}_{\mathbb{T}} : F^{\mathbb{T}'} \rightarrowtail F^{\mathbb{T}}$ and, by Theorem II.34, the theory $\mathbb{T}'$ yields a subtopos $e^{\mathbb{T}'}_{\mathbb{T}} : \mathbf{Sh}\left(F^{\mathbb{T}'}\right) \rightarrowtail \mathbf{Sh}\left(F^{\mathbb{T}}\right) \simeq \mathcal{E}_{\mathbb{T}}$.

For the converse, by Theorem II.34, a subtopos

$$f\colon \mathcal{F} \hookrightarrow \mathbf{Sh}\,(F^{\mathbb{T}}) \simeq \mathcal{E}_{\mathbb{T}}$$

must be induced by an internal locale morphism, i.e. an internal locale morphism $\mathfrak{f}\colon \mathbb{L} \to F^{\mathbb{T}}$ where, for each $\vec{x}$, the component frame homomorphism $\mathfrak{f}_{\vec{x}}^{-1}\colon F^{\mathbb{T}}(\vec{x}) \to \mathbb{L}(\vec{x})$ is surjective. Let $\mathbb{T}_f$ be the quotient theory of $\mathbb{T}$ whose axioms consist of the sequents $\varphi \vdash_{\vec{x}} \psi$, for each pair of formulae φ, ψ for which $\mathfrak{f}_{\vec{x}}^{-1}([\varphi]_{\mathbb{T}}) \leqslant \mathfrak{f}_{\vec{x}}^{-1}([\psi]_{\mathbb{T}})$. To complete the proof, we need only note that $\mathbb{L} \cong F^{\mathbb{T}_f}$ and that $\mathbb{T}_{e^{\mathbb{T}'}_{\mathbb{T}}} \equiv_s \mathbb{T}'$ for each quotient theory of $\mathbb{T}$ and subtopos $f\colon \mathcal{F} \rightarrowtail \mathcal{E}_{\mathbb{T}}$. □

The frame of quotient theories. Given two quotient theories $\mathbb{T}'$, $\mathbb{T}''$ of $\mathbb{T}$, we order the theories by $\mathbb{T}'' \leqslant \mathbb{T}'$ if $\mathbb{T}''$ proves every axiom of $\mathbb{T}'$. Thus, we have that $\mathbb{T} \equiv_s \mathbb{T}''$ if and only if $\mathbb{T}' \leqslant \mathbb{T}''$ and $\mathbb{T}'' \leqslant \mathbb{T}'$. Thus, we are able to form $\mathfrak{Th}_{\mathbb{T}}$, the *poset of quotient theories* of $\mathbb{T}$, whose elements are the $\equiv_s$-equivalence classes of quotient theories of $\mathbb{T}$ ordered by as above.

Corollary III.50 (Theorem 4.1.3 [22])**.** *The poset $\mathfrak{Th}_{\mathbb{T}}$ of quotient theories of $\mathbb{T}$ is a co-frame.*

Proof. We first note that there is a factorisation of internal sublocale embeddings

$$\begin{tikzcd}
F^{\mathbb{T}''} \arrow[rr, tail, "e^{\mathbb{T}''}_{\mathbb{T}'}"] \arrow[dr, tail, "e^{\mathbb{T}''}_{\mathbb{T}}"'] & & F^{\mathbb{T}'} \arrow[dl, tail, "e^{\mathbb{T}'}_{\mathbb{T}}"] \\
& F^{\mathbb{T}} &
\end{tikzcd}$$

if and only if $\mathbb{T}'' \leqslant \mathbb{T}'$. Thus, in combination with Corollary III.49, we have that $\mathfrak{Th}_{\mathbb{T}} \cong N\,(F^{\mathbb{T}})^{\mathrm{op}}$. An application of Theorem II.42 now yields the result. □

Chapter IV

The geometric completion

Completions of doctrines. In the categorical formulation of logic and syntax afforded by doctrine theory, a natural question arises: given a doctrine

$$P: C^{\mathrm{op}} \longrightarrow \mathbf{PreOrd}$$

and a certain syntax we wish P to interpret, is there a universal way of completing P to this new syntax?

Many such logical completions have been studied in recent years. In [96], Pasquali constructs a co-free completion of a primary doctrine to an elementary doctrine. The quotient completion of an elementary doctrine has been extensively studied by Maietti and Rosolini (see [81]–[84]). The existential completion, introduced by Trotta in [119], universally completes a primary doctrine to an existential doctrine. The existential completion is adapted by Trotta and Spadetto in [121, §3] to give a completion of a primary doctrine to one which interprets universal quantification. In [30] Coumans gives a completion of coherent doctrines that generalises the canonical extension of distributive lattices.

The geometric completion we present is another such completion.

Philosophical motivation. While syntactic completions of doctrines are obviously of a philosophical interest for their universal property, it is also desirable that they be *semantically invariant*, i.e. the category of models associated with a doctrine P and its completion TP are categorically equivalent. Thus, one is able to study the semantics of the doctrine P but within the potentially more familiar framework of the syntax of TP. We will observe in Theorem IV.16 that, if the desired models of a doctrine P are encoded by a Grothendieck topology J on $C \rtimes P$, then the geometric completion of (P, J) is semantically invariant. It is this property which allows the intended use of the geometric completion: to re-express a study of the semantics for various doctrines by a single treatment for geometric doctrines.

The ideal completion for preorders. The geometric completion we will study is a fibred generalisation of the ideal completion for preorders, which is described in [68, §III.4] and [60, §II.2.11]. Recall that given a preorder P, the preorder $2^{P^{\mathrm{op}}}$ of all monotone maps $f: P^{\mathrm{op}} \to 2$, given their pointwise ordering, is the *free join completion* of P (the universal property of $2^{P^{\mathrm{op}}}$ can be deduced from Remark IV.2). Furthermore,

$2^{P^{\mathrm{op}}}$ is isomorphic to the set of down-sets of P ordered by inclusion, and is additionally a frame.

Recall also that if we endow P with a Grothendieck topology (a Grothendieck topology on a preorder is sometimes also called a *covering system*), then we can form the frame J-$\mathbf{Idl}(P)$ of *J-ideals* on P. The elements of J-$\mathbf{Idl}(P)$ are the J-closed down-sets $I \subseteq P$, i.e. those down-sets such that if $\{ y_j \leqslant x \mid j \in J \}$ is a J-covering sieve with each y_j in I, then $x \in I$ too.

The map $\eta^{(P,J)}\colon P \to J\text{-}\mathbf{Idl}(P)$, which sends an element $x \in P$ to the J-closure of the principal down-set $\downarrow x$ generated by x, constitutes a 'geometric completion' of the pair (P, J) in the sense that it satisfies a universal property:

Theorem IV.1 (Proposition II.2.11 [60])**.** *For a meet-semilattice P and a Grothendieck topology J on P, the frame J-$\mathbf{Idl}(P)$ satisfies the universal property that for each meet-semilattice homomorphism $a\colon P \to L$ into a frame L which is* J-continuous*, meaning that $a(x) = \bigvee_{y \leqslant x \in S} a(y)$ for each J-covering sieve S on x, there exists a unique frame homomorphism $\mathfrak{a}\colon J\text{-}\mathbf{Idl}(P) \to L$ for which the triangle*

$$\begin{tikzcd} P \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "a"'] & J\text{-}\mathbf{Idl}(P) \arrow[d, dashed, "\mathfrak{a}"] \\ & L \end{tikzcd}$$

commutes.

Remark IV.2 (Theorem 6.2 [19])**.** In Theorem IV.1, the requirement that P be a meet-semilattice can be relaxed. If P is any preorder, and J is a Grothendieck topology on P, then the frame J-$\mathbf{Idl}(P)$ satisfies the universal property that, for each J-continuous monotone map $a\colon P \to L$ into a frame L, there exists a unique monotone map $\mathfrak{a}\colon J\text{-}\mathbf{Idl}(P) \to L$ such that $\mathfrak{a}$ preserves all joins and $\mathfrak{a} \circ \eta^{(P,J)} = a$.

The map $\mathfrak{a}$ also preserves finite meets, and hence is a frame homomorphism, if and only if $a\colon P \to L$ defines a morphism of sites $a\colon (P, J) \to (L, J_{\mathrm{can}})$.

For certain cases, the geometric completion can be understood as an *internal ideal completion*, externalised via the perspective of fibred topos theory [24, §6]. A strictly functorial doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$, fibred over a *small* category $\mathcal{C}$, can be viewed as an *internal preorder* of the presheaf topos $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, and a Grothendieck topology J on the Grothendieck construction $\mathcal{C} \rtimes P$ acts as an internal covering system on P. The pair (P, J) admits a *fibred ideal completion*, as established in [24, Theorem 6.1], that generalises the ideal completion for preorders. This yields the geometric completion of the doctrine P, relative to the Grothendieck topology J on $\mathcal{C} \rtimes P$.

Key results. The universal property of the geometric completion we present in Theorem IV.16 is an extension (to include change of base category) of the universal property of the fibred ideal completion established in [24]. We will show that the geometric completion defines an idempotent 2-monad on the category of doctrinal sites **DocSites** from Definitions III.22. We also simplify the description of the geometric completion from [24] for certain doctrines. This simpler description can be leveraged to recover the geometric completion on an arbitrary doctrine.

We also relate the geometric completion to two other classes of completions of doctrines. The first are *coarse geometric completions*, which are obtained when we 'forget' some of the geometric information added by geometric completion. This is encoded by equipping the geometric completion of a doctrinal site with a weaker topology. The coarse geometric completions thus obtained are no longer idempotent but are instead *lax-idempotent*.

The latter class we study are *subgeometric completions*, which are intended to capture completions of doctrines to some, but not all, of the data of geometric syntax – for example, in Section IV.3.3 we will prove that Trotta's existential completion [119] is subgeometric.

We also demonstrate throughout how these completions of doctrines yield completions of categories, such as the *regular completion* [27], via the syntactic category construction from Section III.3.

Overview. The chapter is divided as follows.

(A) The geometric completion is developed in Section IV.1 as an application of the fibred ideal completion of [24, §6]. That the geometric completion of a doctrinal site is universal, semantically invariant and idempotent is proved in Section IV.1.2, extending the universal property found in [24], and this universal property is used to develop the 2-monadic aspects of the geometric completion in Section IV.1.3.

We also describe how, in combination with the syntactic category construction from Section III.3, the geometric completion for doctrines yields a geometric completion for regular sites in Section IV.1.4.

(B) The geometric completion is idempotent since we can keep track of the geometricity of a geometric doctrine by assigning a suitable Grothendieck topology. Section IV.2 is dedicated to the study of completions when some of this information is 'forgotten' by assigning a *coarser* Grothendieck topology. We develop a general framework for *coarse geometric completions*, and also prove that every coarse geometric completion is lax-idempotent.

(C) Finally, we study *subgeometric completions* in Section IV.3. Vaguely speaking, a 2-monad T on a 2-subcategory of doctrines, viewed as a completion of doctrines at the suggestion of [120], is 'subgeometric' if a suitable sub-class of geometric doctrines are all T-algebras and the data added to the completion TP of a doctrine P can be 'seen' by a certain Grothendieck topology J_P^T on $\mathcal{D} \rtimes TP$. We will show that the geometric completion of P is isomorphic to the doctrine obtained by completing P according to T, keeping track of the new data by the topology J_P^T, and then geometrically completing.

We begin in Section IV.3.1 with a motivating example: Trotta's existential completion [119]. The formal definition of a subgeometric completion is introduced in Section IV.3.2, where we also give sufficient conditions for a subgeometric completion to be lax-idempotent.

We then turn to further examples of subgeometric completions. In Section IV.3.3 we develop a general theory for obtaining subgeometric completions via subdoctrines of the free geometric completion, which encompasses the existential completion and the coherent completion of a primary doctrine. We also relate these completions to the *regular* and *coherent* completion of a cartesian category (see [27]). Finally, various pointwise completions are shown to be subgeometric in Section IV.3.4.

IV.1 The geometric completion of a doctrine

We are able to define the geometric completion of a doctrinal site using only results on internal locales recalled in Chapter II. Given a doctrinal site $(P, J) \in \mathbf{DocSites}$ the relative topos

$$C_{\pi_P} \colon \mathbf{Sh}(C \rtimes P, J) \longrightarrow \mathbf{Sets}^{C^{\mathrm{op}}}$$

is localic by [23, Proposition 7.11] and the fact that π_P is a faithful functor (alternatively, C_{π_P} is localic by [63, Examples A4.6.2(a) & (c)]). Thus, by Theorem II.7, the topos $\mathbf{Sh}(C \rtimes P, J)$ is the topos of sheaves on an internal locale (i.e. a geometric doctrine)

$$C_{\pi_{P*}}(\Omega_{\mathbf{Sh}(C \rtimes P, J)}) \colon C^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}}$$

of $\mathbf{Sets}^{C^{\mathrm{op}}}$.

Definition IV.3 (Definition 6.2 [24])**.** The *geometric completion* of a doctrinal site (P, J) is the geometric doctrine $C_{p_{P*}}(\Omega_{\mathbf{Sh}(C \rtimes P, J)}) \colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$. We denote the geometric completion of (P, J) by $\mathfrak{z}(P, J)$.

Recall from [24, Proposition 4.2] or Section II.2 that the doctrine

$$\mathfrak{z}(P, J) \colon C^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}}$$

is isomorphic to the functor

$$\mathrm{Sub}_{\mathbf{Sh}(C \rtimes P, J)}(C^*_{p_P} \circ \text{よ}_C(-)) \colon C^{\mathrm{op}} \longrightarrow \mathbf{Frm}_{\mathrm{open}}.$$

We claim that this choice of geometric completion is 2-functorial in **DocSites**, and moreover universal, idempotent and semantically invariant. The proof of these facts is delayed until Section IV.1.2 and Section IV.1.3. We proceed as follows.

- Immediately below, in Section IV.1.1, we recall the explicit description of the geometric completion $\mathfrak{z}(P, J)$ of a doctrinal site (P, J), as described in [24, §6]. We also demonstrate that, in special cases, the calculation of the geometric completion can be simplified.

  Firstly, we show that the geometric completion of an existential doctrinal site (seen in Section III.3.1) can be computed 'pointwise'. Secondly, we give a simpler description of the geometric completion in the case where each fibre of P has a top element and these are preserved by transition maps. By showing that the *free top completion* is subgeometric, we can recover the geometric completion of an arbitrary doctrine.
- In Section IV.1.2, the unit of the geometric completion is defined and the universal property is proved.

– We demonstrate the 2-monadic aspects of the geometric completion in Section IV.1.3 and identify the algebras of the monad as the geometric doctrines.
– Finally, in Section IV.1.4, we relate the geometric completion for doctrines to the geometric completion for regular sites via the syntactic category construction from Section III.3.

IV.1.1 Calculating the geometric completion

An explicit description of the geometric completion $\mathfrak{Z}(P, J) \colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ for a doctrinal site (P, J) can be computed directly using the description of the subobject classifier of a Grothendieck topos found in [79, §III.7], as is done in [24, Proposition 6.2]. This returns $\mathfrak{Z}(P, J)$ as the doctrine where:

(i) for each object $c \in C$, $\mathfrak{Z}(P, J)(c)$ is the frame of *J-closed subobjects* in $\mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}}$ of the presheaf $C(\pi_P(-), c) \colon (C \rtimes P)^{\mathrm{op}} \to \mathbf{Sets}$ (for a description of J-closed subobjects, see [23, §2.1]),

(ii) and for each arrow $d \xrightarrow{f} c$ of C, the transition map

$$\mathfrak{Z}(P, J)(f) \colon \mathfrak{Z}(P, J)(c) \longrightarrow \mathfrak{Z}(P, J)(d)$$

sends a J-closed subobject $\varsigma \rightarrowtail C(\pi_P(-), c)$ to the pullback

$$\begin{tikzcd} f^*(\varsigma) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \varsigma \arrow[d, tail] \\ C(\pi_P(-), d) \arrow[r] & C(\pi_P(-), c). \end{tikzcd}$$

By unravelling definitions, this is equivalent to the concrete description presented below.

Construction IV.4. Let $P \colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine and let J a Grothendieck topology on $C \rtimes P$. The geometric completion $\mathfrak{Z}(P, J) \colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ admits the following description.

(i) For each object c of C, an element S of $\mathfrak{Z}(P, J)(c)$ is a set of pairs (f, x), where $d \xrightarrow{f} c$ is an arrow of C and $x \in P(d)$, such that:

a) if $(f, x) \in S$, then $(f \circ g, y) \in S$ for each arrow $e \xrightarrow{g} d$ of C and $y \in P(e)$ with $y \leqslant P(g)(x)$;

b) for each arrow $d \xrightarrow{f} c$ of C, given a subset $\{ (g_i, y_i) \mid i \in I \} \subseteq S$ such that, for each $i \in I$, g_i factors as

$$\begin{tikzcd} e_i \arrow[d, "h_i"'] \arrow[dr, "g_i"] & \\ d \arrow[r, "f"] & c, \end{tikzcd}$$

if there is an $x \in P(d)$ and, for all $i \in I$, $y_i \leqslant P(h_i)(x)$ for which the family

$$\left\{ (e_i, y_i) \xrightarrow{h_i} (d, x) \,\middle|\, i \in I \right\}$$

of morphisms in $C \rtimes P$ is J-covering, then $(f, x) \in S$.

We then order $\mathfrak{Z}(P,J)(c)$ by inclusion.

(ii) For each arrow $d \xrightarrow{f} c$ of C, $\mathfrak{Z}(P,J)(f) \colon \mathfrak{Z}(P,J)(c) \to \mathfrak{Z}(P,J)(d)$ sends $S \in \mathfrak{Z}(P,J)(c)$ to $f^*(S)$, where

$$f^*(S) = \{\, (g,y) \,\big|\, (f \circ g, y) \in S \,\} \in \mathfrak{Z}(P,J)(d).$$

The closure operator. Clearly, if J and J' are Grothendieck topologies on $C \rtimes P$ with $J' \subseteq J$, then $\mathfrak{Z}(P,J)(c) \subseteq \mathfrak{Z}(P,J')(c)$ for each object c of C. Hence, for every Grothendieck topology J, $\mathfrak{Z}(P,J)(c)$ is a subset of $\mathfrak{Z}(P,J_{\mathrm{triv}})(c)$, where J_{triv} is the trivial topology on $C \rtimes P$. An element $S \in \mathfrak{Z}(P,J_{\mathrm{triv}})(c)$ that is contained in the subset $\mathfrak{Z}(P,J)(c) \subseteq \mathfrak{Z}(P,J_{\mathrm{triv}})(c)$, i.e. S satisfies property (b) above, is said to be *J-closed*. This is precisely what it means for the subobject $\varsigma \rightarrowtail C(\pi_P(-),c)$ corresponding to S to be J-closed in the sense of [23].

A *closure operation for subobjects* is described in [23, §2.1]. In the particular case of subobjects of the presheaf $C(\pi_P(-),c)$, i.e. elements $S \in \mathfrak{Z}(P,J)(c)$, the J-closure can be understood entirely in terms of internal locale theory.

Since the embedding $\mathbf{Sh}(C \rtimes P, J) \rightarrowtail \mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}}$ is a geometric morphism for which the triangle

$$\begin{tikzcd}
\mathbf{Sh}(\mathfrak{Z}(P,J)) \simeq \mathbf{Sh}(C \rtimes P, J) \arrow[rr, tail] \arrow[dr, bend right] & & \mathbf{Sets}^{(C \rtimes P)^{\mathrm{op}}} \simeq \mathbf{Sh}(\mathfrak{Z}(P,J_{\mathrm{triv}})) \arrow[dl, bend left] \\
& \mathbf{Sets}^{C^{\mathrm{op}}} &
\end{tikzcd}$$

commutes, by Theorem II.34 the geometric morphism is induced by an embedding of internal locales $\mathfrak{Z}(P,J) \rightarrowtail \mathfrak{Z}(P,J_{\mathrm{triv}})$. That is, for each object $c \in C$, there is a surjective frame homomorphism $\overline{(-)}_c \colon \mathfrak{Z}(P,J_{\mathrm{triv}})(c) \twoheadrightarrow \mathfrak{Z}(P,J)(c)$ such that, for each arrow $d \xrightarrow{g} c \in C$, the diagram

$$\begin{tikzcd}
\mathfrak{Z}(P,J_{\mathrm{triv}})(d) \arrow[r, shift left, "\exists_{\mathfrak{Z}(P,J_{\mathrm{triv}})(f)}"] \arrow[d, two heads, "\overline{(-)}_d"'] & \mathfrak{Z}(P,J_{\mathrm{triv}})(c) \arrow[l, shift left, "\mathfrak{Z}(P,J_{\mathrm{triv}})(f)"] \arrow[d, two heads, "\overline{(-)}_c"] \\
\mathfrak{Z}(P,J)(d) \arrow[r, shift left, "\exists_{\mathfrak{Z}(P,J)(c)}"] & \mathfrak{Z}(P,J)(c) \arrow[l, shift left, "\mathfrak{Z}(P,J)(f)"]
\end{tikzcd}$$

is a morphism of adjunctions.

Definition IV.5. Let S be an element of $\mathfrak{Z}(P,J_{\mathrm{triv}})(c)$. We call the image of S under

$$\overline{(-)}_c \colon \mathfrak{Z}(P,J_{\mathrm{triv}})(c) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{Z}(P,J)(c)$$

the *J-closure* of S, and denote it by $\overline{S}$. The corresponding subobject $\overline{\varsigma}$ of $C(p_P(-),c)$ is precisely the J-closure of ς.

The geometric completion of an existential doctrinal site. Recall Definition III.27, that an existential doctrinal site $(P,(J_c)_{c \in C})$ consists of a doctrine $P \colon C^{\mathrm{op}} \to \mathbf{PreOrd}$

and a Grothendieck topology J_c on each fibre $P(c)$ such that $J_\rtimes$ defines a Grothendieck topology on $C \rtimes P$, where a sieve

$$\left\{ (c_i, x_i) \xrightarrow{f_i} (d, y) \,\middle|\, i \in I \right\}$$

is $J_\rtimes$-covering if and only if

$$\left\{ (d, \exists_{f_i} x_i) \xrightarrow{\mathrm{id}_d} (d, y) \,\middle|\, i \in I \right\}$$

is J_d-covering, where $\exists_{f_i}$ is a left adjoint to $P(f_i)$ that preserves covers.

Since the arrow $(d, x) \xrightarrow{f} (c, \exists_f x)$ is a $J_\rtimes$-cover for any arrow $d \xrightarrow{f} c \in C$, an element $S \in \mathfrak{Z}(P, J)(c)$ is entirely determined by its elements of the form $(\mathrm{id}_c, x) \in S$. Hence, by Construction IV.4, we obtain the following.

Proposition IV.6. *For an existential doctrinal site $(P, (J_c)_{c \in C})$, the geometric completion $\mathfrak{Z}(P, J_\rtimes)$ is isomorphic to its pointwise ideal completion, that is:*

(i) for each object c of C, $\mathfrak{Z}(P, J)(c)$ is the frame J_c-$\mathbf{Idl}(P(c))$,

(ii) for each arrow $d \xrightarrow{f} c$ of C, $\mathfrak{Z}(P, J)(f) \colon \mathfrak{Z}(P, J)(c) \to \mathfrak{Z}(P, J)(d)$ sends a J_c-ideal I to the J_d-ideal

$$f^*(I) = \left\{ y \in P(d) \,\middle|\, \exists_f y \in I \right\}.$$

Remark IV.7. Recall that existential doctrinal sites were intended to interpret theories that interpret at least the syntax of regular logic, i.e. the symbols $\{\wedge, \top, \exists\}$, if not further syntax. It is therefore not surprising that completing to geometric logic, whose permissible symbols are $\{\wedge, \top, \exists, \bot, \bigvee\}$, involves adding only fibre-wise structure.

Example IV.8. When $P \colon C^{\mathrm{op}} \to \mathbf{DLat}$ is a coherent doctrine, and $C \rtimes P$ is equipped with the topology J_{Coh}, we recognise by Proposition IV.6 that the fibre of the geometric completion $\mathfrak{Z}(P, J_{\mathrm{Coh}})(c)$ is the coherent locale associated with the distributive lattice $P(c)$ under the (point-free) Stone duality for distributive lattices (see [60, §II.3.3], cf. [111]).

In particular, if $\mathbb{B} \colon C^{\mathrm{op}} \to \mathbf{Bool}$ is a Boolean doctrine, then

$$\mathfrak{Z}(\mathbb{B}, J_{\mathrm{Coh}}) \colon C^{\mathrm{op}} \longrightarrow \mathbf{StFrm}_{\mathrm{open}}$$

sends $c \in C$ to the Stone frame corresponding to the Boolean algebra $\mathbb{B}(c)$. If there is an isomorphism $\mathbb{B} \cong F^{\mathbb{T}}$ for some single-sorted classical theory $\mathbb{T}$ over a signature Σ, then $\mathfrak{Z}(\mathbb{B}, J_{\mathrm{Coh}})(\vec{x})$ (where $\vec{x} \in \mathbf{Con}_1$ is a context/tuple of variables of length n) coincides with the frame of opens of the familiar nth Stone space of the theory $\mathbb{T}$ (see [52, §6.3]). Doctrines of this form – or rather, since Stone frames are spatial, the doctrines $\mathrm{Pt} \circ \mathfrak{Z}(\mathbb{B}, K_{\mathbb{B}}^{\mathrm{fin}}) \colon \mathbf{Con}_1^{\mathrm{op}} \to \mathbf{StSpace}$ – were dubbed *polyadic spaces* in the note [64] and suggested for use in categorically proving standard theorems of classical logic, a desire realised in [43].

Examples IV.9. (i) Suppose that $\mathbb{T}$ is a coherent theory (or indeed any subfragment of geometric logic). Then $\mathbb{T}$ can also be considered as a geometric theory. The geometric completion of the doctrinal site $(F^{\mathbb{T}}_{\mathrm{Coh}}, J_{\mathrm{Coh}})$ is simply the geometric doctrine $F^{\mathbb{T}}_{\mathrm{Geom}}$.

(ii) (Morleyization) If $\mathbb{T}$ is instead a classical theory, then by a well-known trick known as *Morleyization* (see [63, Lemma D1.5.13]), there exists a Morita equivalent *geometric* theory $\mathbb{T}'$. The geometric completion of the doctrinal site $(F^{\mathbb{T}}_{\mathrm{Bool}}, J_{\mathrm{Coh}})$ is given by $F^{\mathbb{T}'}_{\mathrm{Geom}}$.

When top elements are available

In the remainder of this subsection we demonstrate that, in the special case of a doctrine $P \colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ where $P(c)$ has a top element, for each object $c \in C$, and $P(f)$ preserves that top element for each arrow $d \xrightarrow{f} c$ of C, the description of the geometric completion given in Construction IV.4 can be simplified further.

We will then show how the description in Construction IV.4 can be recovered for an arbitrary doctrinal site (P, J) by freely adding (preserved) top elements to the doctrine P. This yields a different proof to [24], though more circumlocutory, that Construction IV.4 describes the geometric completion of (P, J). We do so to illustrate our first example of a *subgeometric completion*. A subgeometric completion is a partial completion to the data of a geometric doctrine which can be 'subsumed' by the geometric completion. This will be explored further in Section IV.3. To avoid confusion in the subsequent paragraphs, we will temporarily relabel the doctrine described in Construction IV.4 as $\mathfrak{Z}'(P, J)$ while we prove the isomorphism $\mathfrak{Z}(P, J) \cong \mathfrak{Z}'(P, J)$.

If, for each object c of C, $P(c)$ has a top element $\top_c$ which is preserved by $P(f)$ for each arrow $d \xrightarrow{f} c$ of C, then the projection $\pi_P \colon C \rtimes P \to C$ has a right adjoint: the functor

$$t_P \colon C \to C \rtimes P$$

which sends $c \in C$ to $(c, \top_c) \in C \rtimes P$. Thus, we can apply the description of the direct image of C_{p_P} given in [79, Theorem VII.10.4], to obtain that

$$\mathfrak{Z}(P, J) \cong C_{p_{P*}}(\Omega_{\mathbf{Sh}(C \rtimes P, J)}) = \Omega_{\mathbf{Sh}(C \rtimes P, J)} \circ t_P^{\mathrm{op}} \colon C^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}.$$

Therefore, using the description of the subobject classifier of $\mathbf{Sh}(C \rtimes P, J)$ found in [79, §III.7], for each $c \in C$, an element of $\mathfrak{Z}(P, J)(c)$ is a J-closed sieve S on $(c, \top_c)$ and, for each $d \xrightarrow{f} c \in C$, $\mathfrak{Z}(P, J)(f)$ sends S to

$$f^*(S) = \left\{ (e, V) \xrightarrow{g} (d, \top_d) \,\middle|\, (e, V) \xrightarrow{f \circ g} (c, \top_c) \in S \right\}.$$

We therefore observe that $\mathfrak{Z}(P, J)$ is indeed isomorphic to the doctrine $\mathfrak{Z}'(P, J)$ as described in Construction IV.4. The witnessing isomorphism is given by sending a J-closed sieve S on $(c, \top_c)$ to the set

$$\left\{ (f, x) \,\middle|\, (d, x) \xrightarrow{f} (c, \top_c) \in S \right\} \in \mathfrak{Z}'(P, J)(c).$$

The free top completion. In the absence of top elements, we can freely add them to the doctrine $P \colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ and demonstrate that, by carefully selecting a Grothendieck topology, we obtain a doctrinal site whose geometric completion is isomorphic to $\mathfrak{Z}(P, J)$ – that is to say, adding top elements is a subgeometric completion.

Definition IV.10. Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine and let J be a Grothendieck topology on $\mathcal{C} \rtimes P$.

(i) Denote by $P^{\top}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ the *free top completion*, the doctrine where

a) for each object c of $\mathcal{C}$, $P^{\top}(c)$ is the preorder $P(c) \oplus \top_c$, where a top element $\top_c$ has been freely added to $P(c)$;

b) for each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, $P^{\top}(f)\colon P^{\top}(c) \to P^{\top}(d)$ is the monotone map

$$P^{\top}(f)(x) = \begin{cases} P(f)(x) & \text{if } x \in P(c), \\ \top_d & \text{if } x = \top_c. \end{cases}$$

(ii) We define a Grothendieck topology $J^{\top}$ on $\mathcal{C} \rtimes P^{\top}$ in the following way:

a) for each object of the form (c, x), with $x \in P(c)$, a sieve S on (c, x) is $J^{\top}$-covering if and only if S is J-covering;

b) for an object of the form $(c, \top_c)$, a sieve S on $(c, \top_c)$ is $J^{\top}$-covering if and only if, for each arrow of the form $(d, x) \xrightarrow{f} (c, \top_c)$, the sieve $f^*(S)$ on (d, x) is J-covering.

The terminology free top completion is justified as a universal property is clearly satisfied. For any morphism of doctrines $(F, a)\colon P \to Q$ where $Q(d)$ has a top element for each $c \in \mathcal{C}$ which is preserved by $Q(g)$ for each $e \xrightarrow{g} d \in \mathcal{D}$, there is a unique natural transformation $a^{\top}\colon P^{\top} \Rightarrow Q \circ F^{\mathrm{op}}$ such that the triangle

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr, "a"'] & P^{\top} \arrow[d, dashed, "a^{\top}"] \\ & Q \circ F^{\mathrm{op}} \end{tikzcd}$$

commutes and $a^{\top}_c$ sends $\top_c \in P^{\top}(c)$ to the top element of $Q(F(c))$, for each $c \in \mathcal{C}$.

Note that $\mathcal{C} \rtimes P$ defines a subcategory of $\mathcal{C} \rtimes P^{\top}$ and J is the restriction of $J^{\top}$ to this subcategory. Note also that, for each $c \in \mathcal{C}$, the family

$$\left\{ (d, x) \xrightarrow{f} (c, \top_c) \;\middle|\; x \in P(d), d \xrightarrow{f} c \in \mathcal{C} \right\}$$

generates a $J^{\top}$-covering sieve.

Lemma IV.11. *For each doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to$ **PreOrd** and Grothendieck topology J on $\mathcal{C} \rtimes P$, $J^{\top}$ is a Grothendieck topology on $\mathcal{C} \rtimes P^{\top}$.*

Proof. The maximality and stability conditions for $J^{\top}$ are trivially satisfied since J is a Grothendieck topology on $\mathcal{C} \rtimes P$, as is the transitivity condition $J^{\top}$ for sieves on objects of the form (d, x) with $x \in P(d)$.

It thus remains to show that if S is a $J^{\top}$-covering sieve on $(c, \top_c)$ and R is a sieve on $(c, \top_c)$ such that $h^*(R) \in J^{\top}(d, x)$ for each arrow $(e, y) \xrightarrow{h} (c, \top_c)$ in S, then R is $J^{\top}$-covering, i.e. $f^*(R) \in J(d, x)$ for each arrow $(d, x) \xrightarrow{f} (c, \top_c)$ with $x \in P(d)$.

As $f^*(S) \in J(d,x)$ and, for each arrow $(e,y) \xrightarrow{k} (d,x)$ of $f^*(S)$, i.e. for which the composite $(e,y) \xrightarrow{k} (d,x) \xrightarrow{f} (c,\top_c)$ is an element of S, we have that $f^*(R)$ is J-covering since $k^*(f^*(R)) = (f \circ k)^*(R)$ is $J^\top$-covering (and so J-covering) by the transitivity condition for J. Thus, by definition, R is $J^\top$-covering. □

Lemma IV.12. *The site $(\mathcal{C} \rtimes P, J)$ is a dense subsite of $(\mathcal{C} \rtimes P^\top, J^\top)$.*

Proof. This is immediate since $\mathcal{C} \rtimes P$ is a full subcategory of $\mathcal{C} \rtimes P^\top$ and the only objects not contained in $\mathcal{C} \rtimes P$, i.e. those objects of the form $(c,\top_c)$, are covered by objects contained in the subcategory. □

The free top completion is subgeometric. Having developed the free top completion for a doctrinal site, we can finally observe that this constitutes a subgeometric completion in the current loose sense that $\mathfrak{Z}(P,J) \cong \mathfrak{Z}(P^\top, J^\top)$ (a formal definition of subgeometricity is provided in Section IV.3.2). As a consequence, we obtain the isomorphism $\mathfrak{Z}(P,J) \cong \mathfrak{Z}'(P,J)$ as desired.

Proposition IV.13. *There is a chain of isomorphisms of doctrines:*

$$\mathfrak{Z}(P,J) \cong \mathfrak{Z}(P^\top, J^\top) \cong \mathfrak{Z}'(P^\top, J^\top) \cong \mathfrak{Z}'(P,J).$$

Proof. That $\mathfrak{Z}(P,J) \cong \mathfrak{Z}(P^\top, J^\top)$ follows since the topoi $\mathbf{Sh}(\mathcal{C} \rtimes P, J)$ and $\mathbf{Sh}(\mathcal{C} \rtimes P^\top, J^\top)$ are equivalent by Lemma IV.12. That $\mathfrak{Z}(P^\top, J^\top) \cong \mathfrak{Z}'(P^\top, J^\top)$ follows as $P^\top$ has (preserved) top elements.

We will sketch the isomorphism between $\mathfrak{Z}'(P^\top, J^\top)$ and $\mathfrak{Z}'(P,J)$. We observe that each $J^\top$-closed sieve S on $(c,\top_c)$ is uniquely determined by the set

$$l_c(S) = \left\{ (f,x) \,\middle|\, (d,x) \xrightarrow{f} (c,\top_c) \in S,\ x \in P(d) \right\} \in \mathfrak{Z}'(P,J).$$

If $l_c(S) = l_c(S')$, then S and S' agree on arrows of the form $(d,x) \xrightarrow{f} (c,\top_c)$, where $x \in P(d) \subseteq P^\top(d)$. Conversely, if $(d,\top_d) \xrightarrow{f} (c,\top_c) \in S$, then both S and S' contain the family

$$R = \left\{ (e,x) \xrightarrow{g} (d,\top_d) \xrightarrow{f} (c,\top_c) \,\middle|\, x \in P(e), e \xrightarrow{g} d \in \mathcal{C} \right\}$$

which covers $(d,\top_d) \xrightarrow{f} (c,\top_c)$. Hence, $(d,\top_d) \xrightarrow{f} (c,\top_c) \in S'$ too. The same argument with S and S' swapped completes the proof that $l_c(S) = l_c(S')$ implies that $S = S'$. The maps l_c, for each $c \in \mathcal{C}$, are thus evidently the components of a natural isomorphism between $\mathfrak{Z}'(P^\top, J^\top)$ and $\mathfrak{Z}'(P,J)$. □

IV.1.2 Universal property of the geometric completion

We are now able to prove that the geometric completion of a doctrinal site is universal in **DocSites**, idempotent and semantically invariant as claimed. We first recall the construction of the unit of the geometric completion, which, unsurprisingly, is the same as the unit of the fibred ideal completion defined in [24, Proposition 6.2], before turning to the universal property of the geometric completion, which extends the universal property of [24]. Finally, we discuss some of the basic preservation properties of the unit.

The unit of the geometric completion. The unit generalises the notion of taking the closure of a principal down-set for a preorder with a covering system. Let P be a doctrine and let J be a Grothendieck topology on $\mathcal{C} \rtimes P$. For each object $c \in \mathcal{C}$ and $x \in P(c)$, the set

$$\downarrow x = \left\{ (g, y) \,\middle|\, e \xrightarrow{g} c \in \mathcal{C},\ y \in P(e) \text{ and } y \leqslant P(g)(x) \right\}$$

is an object of $\mathfrak{Z}(P, J_{\mathrm{triv}})(c)$. For each arrow $d \xrightarrow{f} c$, the transition map

$$\mathfrak{Z}(P, J_{\mathrm{triv}})(f) \colon \mathfrak{Z}(P, J_{\mathrm{triv}})(c) \longrightarrow \mathfrak{Z}(P, J_{\mathrm{triv}})(d)$$

acts on this element by

$$\begin{aligned} \mathfrak{Z}(P, J_{\mathrm{triv}})(f)(\downarrow x) &= \{ (h, y) \mid (f \circ h, y) \in \downarrow x \}, \\ &= \left\{ (h, y) \,\middle|\, e \xrightarrow{h} c \in \mathcal{C},\ y \in P(e) \text{ and } y \leqslant P(f \circ g)(x) \right\}, \\ &= \left\{ (h, y) \,\middle|\, e \xrightarrow{h} c \in \mathcal{C},\ y \in P(e) \text{ and } y \leqslant P(g)P(f)(x) \right\}, \\ &= \downarrow P(f)(x). \end{aligned}$$

Hence, we obtain a (pseudo-)natural transformation $\downarrow(-) \colon P \Rightarrow \mathfrak{Z}(P, J_{\mathrm{triv}})$.

Definition IV.14. Let (P, J) be a doctrinal site. We will use

$$\eta^{(P,J)} \colon P \Rightarrow \mathfrak{Z}(P, J)$$

to denote the the composite (pseudo-)natural transformation

$$P \xRightarrow{\downarrow(-)} \mathfrak{Z}(P, J_{\mathrm{triv}}) \xRightarrow{\overline{(-)}} \mathfrak{Z}(P, J).$$

The natural transformation $\eta^{(P,J)}$ yields a morphism of doctrinal sites

$$(\mathrm{id}_{\mathcal{C}}, \eta^{(P,J)}) \colon (P, J) \longrightarrow (\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)}).$$

In fact, as shown in [24, Proposition 7.2], one can prove a stronger statement.

Proposition IV.15 (Proposition 7.2 [24]). *The induced functor* $\mathrm{id}_{\mathcal{C}} \rtimes \eta^{(P,J)}$ *yields a dense morphism of sites*

$$\mathrm{id}_{\mathcal{C}} \rtimes \eta^{(P,J)} \colon (\mathcal{C} \rtimes P, J) \longrightarrow (\mathcal{C} \rtimes \mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)}).$$

From Proposition IV.15, we immediately deduce that for each object c of $\mathcal{C}$ and each $S \in \mathfrak{Z}(P, J)(c)$,

$$S = \bigvee_{(f,x) \in S} \exists_{\mathfrak{Z}(P,J)(f)} \left(\eta_d^{(P,J)}(x) \right). \tag{IV.i}$$

We will frequently abuse notation and write $\eta^{(P,J)}$ for the natural transformation, the morphism of doctrinal sites $(\mathrm{id}_{\mathcal{C}}, \eta^{(P,J)}) \colon (P, J) \to (\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)})$, and the functor $\mathrm{id}_{\mathcal{C}} \rtimes \eta^{(P,J)} \colon \mathcal{C} \rtimes P \to \mathcal{C} \rtimes \mathfrak{Z}(P, J)$.

The universal property of the geometric completion.

Theorem IV.16. *To each doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ *and Grothendieck topology* J *on* $\mathcal{C} \rtimes P$*, the natural transformation* $\eta^{(P,J)}\colon P \Rightarrow \mathfrak{Z}(P,J)$ *constitutes the unit of the geometric completion of* (P,J) *for which the following properties are satisfied.*

(i) **Universality:** *for each morphism of doctrinal sites* $(F,a)\colon (P,J) \to (\mathbb{L}, K_{\mathbb{L}})$ *whose codomain is a geometric doctrine* $\mathbb{L}\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$*, there exists a unique morphism of geometric doctrines*

$$(F,\mathfrak{a})\colon \mathfrak{Z}(P,J) \longrightarrow \mathbb{L}$$

such that the triangle

$$\begin{tikzcd} P \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "a"'] & \mathfrak{Z}(P,J) \arrow[d, dashed, "\mathfrak{a}"] \\ & \mathbb{L} \circ F^{\mathrm{op}} \end{tikzcd}$$

commutes;

(ii) **Semantic invariance:** *if the desired (set-based) models P are encoded by J, i.e. there is an equivalence*

$$P\text{-mod}(\mathbf{Sets}) \simeq \mathbf{DocSites}((P,J),(\mathscr{P},K_{\mathscr{P}})),$$

then there is an equivalence of categories of models

$$P\text{-mod}(\mathbf{Sets}) \simeq \mathfrak{Z}(P,J)\text{-mod}(\mathbf{Sets});$$

(iii) **Idempotency:** *for each doctrinal site* (P,J)*, we have that*

$$\mathfrak{Z}(P,J) \cong \mathfrak{Z}(\mathfrak{Z}(P,J), K_{\mathfrak{Z}(P,J)}).$$

Proof. Let $(F,a)\colon (P,J) \to (\mathbb{L},K_{\mathbb{L}})$ be a morphism of doctrinal sites. By Lemma I.19, there exists a commutative square of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{L}) \simeq \mathbf{Sh}(\mathcal{D} \rtimes \mathbb{L}, K_{\mathbb{L}}) \arrow[r, "\mathbf{Sh}(F \rtimes a)"] \arrow[d, "C_{\pi_{\mathbb{L}}}"'] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C} \rtimes P, J) \arrow[d, "C_{\pi_P}"] \\ \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \arrow[r, "\mathbf{Sh}(F)"] & \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}. \end{tikzcd}$$

Let $g\colon \mathbf{Sh}(\mathbb{L}) \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ denote the composite geometric morphism

$$\mathbf{Sh}(\mathbb{L}) \xrightarrow{C_{\pi_{\mathbb{L}}}} \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \xrightarrow{\mathbf{Sh}(F)} \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}.$$

By [24, Proposition 7.2], the factoring topos in the hyperconnected-localic factorisation of g (see [63, §A4.6]) is given by the topos of sheaves on the internal locale $g_*(\Omega_{\mathbf{Sh}(\mathbb{L})})$ of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Whence, we have that

$$\begin{aligned} g_*(\Omega_{\mathbf{Sh}(\mathbb{L})}) &= \mathbf{Sh}(F)_* \circ C_{\pi_{\mathbb{L}}*}(\Omega_{\mathbf{Sh}(\mathbb{L})}), \\ &= C_{\pi_{\mathbb{L}}*}(\Omega_{\mathbf{Sh}(\mathbb{L})}) \circ F^{\mathrm{op}}, \\ &\cong \mathbb{L} \circ F^{\mathrm{op}}. \end{aligned}$$

Therefore, as C_{π_P} is localic, by [63, Lemma A4.6.4] there is a factorisation of $\mathbf{Sh}(F \rtimes a)$ as

$$\begin{tikzcd}
\mathbf{Sh}(\mathbb{L}) \arrow[r] \arrow[d, "C_{\pi_{\mathbb{L}}}"'] \arrow[rr, bend left, "\mathbf{Sh}(F\rtimes a)"] & \mathbf{Sh}(\mathbb{L} \circ F^{\mathrm{op}}) \arrow[r, dashed] \arrow[d, "C_{\pi_{\mathbb{L}\circ F^{\mathrm{op}}}}"] \arrow[dl, phantom, "\cong"] & \mathbf{Sh}(C \rtimes P, J) \arrow[dl, bend left, "C_{\pi_P}"] \\
\mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \arrow[r, "\mathbf{Sh}(F)"] & \mathbf{Sets}^{C^{\mathrm{op}}}. &
\end{tikzcd}$$

Thus, as $\mathbf{Sh}(C \rtimes P, J) \simeq \mathbf{Sh}(\mathfrak{Z}(P, J))$, there is a commutative triangle of geometric morphisms

$$\begin{tikzcd}
\mathbf{Sh}(\mathbb{L} \circ F^{\mathrm{op}}) \arrow[rr] \arrow[dr, "C_{\pi_{\mathbb{L}\circ F^{\mathrm{op}}}}"'] & & \mathbf{Sh}(\mathfrak{Z}(P, J)) \arrow[dl, "C_{\pi_{\mathfrak{Z}(P,J)}}"] \\
& \mathbf{Sets}^{C^{\mathrm{op}}}. &
\end{tikzcd}$$

Therefore, by **??**, we obtain a morphism of internal locales $\mathfrak{a} \colon \mathbb{L} \circ F^{\mathrm{op}} \to \mathfrak{Z}(P, J)$, or rather a morphism of geometric doctrines $(F, \mathfrak{a}) \colon \mathfrak{Z}(P, J) \to \mathbb{L}$, satisfying the required conditions.

That the geometric completion is semantically invariant follows from the relative Diaconescu's equivalence:

$$\begin{aligned}
P\text{-mod}(\mathbf{Sets}) &\simeq \mathbf{DocSites}((P, J), (\mathscr{P}, K_{\mathscr{P}})), \\
&\simeq \mathbf{Geom}(\mathbf{Sets}, \mathbf{Sh}(C \rtimes P, J)), \\
&\simeq \mathbf{Geom}(\mathbf{Sets}, \mathbf{Sh}(C \rtimes \mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)})), \\
&\simeq \mathbf{DocSites}((\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)}), (\mathscr{P}, K_{\mathscr{P}})), \\
&\simeq \mathfrak{Z}(P, J)\text{-mod}(\mathbf{Sets}).
\end{aligned}$$

That the geometric completion is idempotent follows from the fact that

$$\mathfrak{Z}(P, J) \cong C_{\pi_{\mathfrak{Z}(P,J)}*}(\Omega_{\mathbf{Sh}(\mathfrak{Z}(P,J))}) = \mathfrak{Z}(\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)}).$$

□

Remark IV.17. A direct proof of Theorem IV.16, without mention of internal locales, could also be given. Given a morphism of doctrinal sites $(F, a) \colon (P, J) \to (\mathbb{L}, K_{\mathbb{L}})$, where $\mathbb{L} \colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ is a geometric doctrine, we obtain the unique morphism of geometric doctrines $(F, \mathfrak{a}) \colon \mathfrak{Z}(P, J) \to \mathbb{L}$ that makes the triangle

$$\begin{tikzcd}
P \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "a"'] & \mathfrak{Z}(P, J) \arrow[d, dashed, "\mathfrak{a}"] \\
& \mathbb{L} \circ F^{\mathrm{op}}
\end{tikzcd}$$

commute by defining, for each $S \in \mathfrak{Z}(P, J)(c)$,

$$\mathfrak{a}_c(S) = \bigvee_{(g,x) \in S} \exists_{\mathbb{L}(F(g))} a_d(x).$$

Remark IV.18. Given a theory $\mathbb{T}$ over a signature Σ with N sorts, the category $\mathbf{Con}_N$ of contexts is normally considered to be entirely *algebraic* in content. That is to say, the semantics of the empty theory $\mathbb{O}_\Sigma$ over the signature Σ are equivalent to the flat functors $\mathbf{Flat}(\mathbf{Con}_\Sigma, \mathbf{Sets})$ (see [63, Corollary D3.1.2] or Proposition III.5). In order to amplify the analogy with theories, we have elected to work with doctrinal sites (P, J), where only the category $\mathcal{C} \rtimes P$ is endowed with a Grothendieck topology J representing richer syntax, while the base category $\mathcal{C}$ is effectively treated as being endowed with the trivial topology.

We could rectify this myopia by considering the 2-category $\mathbf{DocSites}_{\mathrm{WTB}}$, the 2-category of doctrinal sites *with topologies on the base category*.

(i) The objects of $\mathbf{DocSites}_{\mathrm{WTB}}$ are relative sites

$$[\pi_P \colon (\mathcal{C} \rtimes P, K) \to (\mathcal{C}, J)],$$

where $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ is a doctrine;

(ii) a 1-cell

$$(F, F \rtimes a) \colon (\mathcal{C}, J, \mathcal{C} \rtimes P, K) \longrightarrow (\mathcal{D}, J', \mathcal{D} \rtimes Q, K')$$

of $\mathbf{DocSites}_{\mathrm{WTB}}$ consists of a morphism of doctrines $(F, a) \colon P \to Q$ such that $(F, F \rtimes a)$ is a morphism of relative sites.

(iii) The 2-cells we include are the same as for $\mathbf{DocSites}$.

We note that, because for each object $(\mathcal{C}, J, \mathcal{C} \rtimes P, K) \in \mathbf{DocSites}_{\mathrm{WTB}}$ the geometric morphism

$$C_{\pi_P} \colon \mathbf{Sh}(\mathcal{C} \rtimes P, K) \longrightarrow \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

factors through $\mathbf{Sh}(\mathcal{C}, J) \rightarrowtail \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, by Lemma II.16 the Grothendieck topology $K_{\mathfrak{Z}(P,K)}$ on $\mathcal{C} \rtimes \mathfrak{Z}(P, K)$ contains the Giraud topology $J_{\pi_{\mathfrak{Z}(P,K)}}$, where $\mathfrak{Z}(P, K)$ denotes the geometric completion of (P, K) as in Definition IV.3. Hence, $(\mathcal{C}, J, \mathcal{C} \rtimes \mathfrak{Z}(P, K), K_{\mathfrak{Z}(P,K)})$ defines an object of $\mathbf{DocSites}_{\mathrm{WTB}}$.

Applying a similar method to that employed in Theorem IV.16, we can deduce that

$$(\mathcal{C}, J, \mathcal{C} \rtimes \mathfrak{Z}(P, K), K_{\mathfrak{Z}(P,K)})$$

is the universal completion of $(\mathcal{C}, J, \mathcal{C} \rtimes P, K)$ to an object of $\mathbf{DocSites}_{\mathrm{WTB}}$ of the form $(\mathcal{D}, J', \mathcal{D} \rtimes \mathbb{L}, K_{\mathbb{L}})$ for an internal locale $\mathbb{L} \colon \mathcal{D} \to \mathbf{Frm}_{\mathrm{open}}$ of $\mathbf{Sh}(\mathcal{D}, J')$.

The universal property of the geometric completion as stated in Theorem IV.16 is therefore the restriction of this more general statement to the 1-full 2-subcategory of $\mathbf{DocSites}_{\mathrm{WTB}}$ on objects of the form

$$(\mathcal{C}, J_{\mathrm{triv}}, \mathcal{C} \rtimes P, K),$$

i.e. the 2-category $\mathbf{DocSites}$ from Definitions III.22. However, as explained above, for the purposes of our intended, logical applications the extra generality is not needed.

Preservation properties of the unit. For each object $c \in C$, the unit $\eta_c^{(P,J)} : P \to \mathfrak{Z}(P,J)$ preserves finite meets. This can be seen directly. If the meet $x \wedge y$ of two elements $x, y \in P(c)$ exists, or if $P(c)$ has a top element $\top_c$, then the meet $\downarrow x \wedge \downarrow y \in \mathfrak{Z}(P, J_{\mathrm{triv}})(c)$ is given by $\downarrow(x \wedge y)$ and $\downarrow \top_c$ defines a top element of $\mathfrak{Z}(P, J_{\mathrm{triv}})(c)$. Thus, as

$$\overline{(-)}_c : \mathfrak{Z}(P, J_{\mathrm{triv}}) \longrightarrow \mathfrak{Z}(P,J)$$

preserves finite meets as well, so does the composite $\eta_c^{(P,J)} = \overline{(-)}_c \circ \downarrow(-)_c$.

It is also easily recognised that joins and existential quantifiers are preserved by the unit $\eta^{(P,J)}$ if and only if the Grothendieck topology J is of a certain form. Given a subset $\{ y_i \mid i \in I \} \subseteq P(c)$ whose join $\bigvee_{i \in I} y_i$ exists in $P(c)$, since

$$\mathrm{id}_C \rtimes \eta^{(P,J)} : (C \rtimes P, J) \longrightarrow (C \rtimes \mathfrak{Z}(P,J), K_{\mathfrak{Z}(P,J)})$$

is cover preserving and reflecting by Proposition IV.15, we have that

$$\eta_c^{(P,J)}\left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I} \eta_c^{(P,J)}(y_i)$$

if and only if

$$\left\{ (c, y_i) \xrightarrow{\mathrm{id}_c} \left(c, \bigvee_{i \in I} y_i\right) \,\middle|\, i \in I \right\}$$

is a J-covering family. Identically, if $P(f) : P(c) \to P(d)$ has a left adjoint $\exists_{P(f)}$, then, for each $x \in P(d)$,

$$\eta_c^{(P,J)} \circ \exists_{P(f)}(x) = \exists_{\mathfrak{Z}(P,J)} \circ \eta_d^{(P,J)}(x),$$

if and only if the singleton

$$\left\{ (d, x) \xrightarrow{f} (c, \exists_{P(f)} x) \right\}$$

is a J-covering arrow.

IV.1.3 The geometric completion as a monad

In [119, §5], the language of 2-monad theory is used to describe the universal property of the existential completion. This is expanded upon in [120] into a rich description of the logical completions of elementary doctrines via 2-monad theory. Thus inspired, we will use the language of 2-monad theory for investigating the geometric completion.

Recall that a 2-monad on a 2-category C is a triple (T, η, μ) consisting of an 2-endofunctor $T : C \to C$, and 2-natural transformations $\eta : \mathrm{id}_C \to T$ and $\mu : T \circ T \to T$ such that the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_C \circ T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \circ \mathrm{id}_C \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \tag{IV.ii}$$

strictly commute. The geometric completion will be a 2-monad on the 2-category **DocSites**.

We will initially develop the 1-monadic structure, and add the 2-monadic structure in Proposition IV.19. For any morphism $(F, a)\colon (P, J) \to (Q, K)$ of doctrinal sites, there exists a morphism of geometric doctrinal sites $(F, \mathfrak{a})\colon \mathfrak{Z}(P, J) \to \mathfrak{Z}(Q, K)$ by the universal property of the geometric completion:

$$\begin{tikzcd}
(P, J) \arrow[r, "{(\mathrm{id}_{\mathcal{C}}, \eta^{(P,J)})}"] \arrow[d, "{(F,a)}"'] & (\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)}) \arrow[d, dashed, "{(F,\mathfrak{a})}"] \\
(Q, K) \arrow[r, "{(\mathrm{id}_{\mathcal{D}}, \eta^{(Q,K)})}"] & (\mathfrak{Z}(Q, K), K_{\mathfrak{Z}(Q,K)}).
\end{tikzcd}$$

Thus, the geometric completion is *1-functorial* in that it yields a 1-functor

$$\mathfrak{Z}\colon \mathbf{DocSites} \longrightarrow \mathbf{GeomDoc}.$$

The universal property of the geometric completion ensures that the functor $\mathfrak{Z}$ is a left 1-adjoint to the inclusion of geometric doctrines into doctrinal sites:

$$\begin{tikzcd}
\mathbf{DocSites} \arrow[r, shift left, "\mathfrak{Z}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}. \arrow[l, hook', shift left]
\end{tikzcd} \tag{IV.iii}$$

The unit of the adjunction is the natural transformation $\eta\colon \mathrm{id}_{\mathbf{DocSites}} \to \mathfrak{Z}$ whose component at a doctrinal site (P, J) is $\eta^{(P,J)}\colon (P, J) \to (\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)})$. The counit of the adjunction is the natural transformation whose component at a geometric doctrine $\mathbb{L}$ is the isomorphism of geometric doctrines $\mathbb{L} \cong \mathfrak{Z}(\mathbb{L}, K_{\mathbb{L}})$ induced by the equivalence of topoi $\mathbf{Sh}(\mathbb{L}) \simeq \mathbf{Sh}(\mathfrak{Z}(\mathbb{L}, K_{\mathbb{L}}))$.

In Proposition IV.19 below we add the 2-monadic aspects. The strict 2-adjunction we prove extends the 2-adjunction found in [24, Theorem 7.1], which presents the universal property of the geometric completion without base change (i.e. all doctrines considered are fibred over the same base category).

Proposition IV.19 (cf. Theorem 7.1 [24])**.** *The geometric completion*

$$\mathfrak{Z}\colon \mathbf{DocSites} \longrightarrow \mathbf{GeomDoc}$$

can be made into a 2-functor such that

$$\begin{tikzcd}
\mathbf{DocSites} \arrow[r, shift left, "\mathfrak{Z}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc} \arrow[l, hook', shift left]
\end{tikzcd}$$

is a strict 2-adjunction.

Proof. We first show that $\mathfrak{Z}$ can be made 2-functorial. Let $(F, a), (F', a')\colon (P, J) \rightrightarrows (Q, K)$ be morphisms of doctrinal sites. We must show that every natural transformation $\alpha\colon F \Rightarrow F'$ that defines a 2-cell between morphisms of doctrinal sites

$$\begin{tikzcd}
(P, J) \arrow[r, bend left, "{(F,a)}"] \arrow[r, bend right, "{(F',a')}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (Q, K)
\end{tikzcd}$$

also yields a 2-cell of morphisms of geometric doctrines

$$\begin{tikzcd} \mathfrak{z}(P,J) \arrow[rr, bend left, "{(F,\mathfrak{a})}"] \arrow[rr, bend right, "{(F',\mathfrak{a}')}"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathfrak{z}(Q,K). \end{tikzcd}$$

That is, we must show that, for each $c \in \mathcal{C}$ and $S \in \mathfrak{z}(P,J)(c)$,

$$\mathfrak{a}_c(S) \leqslant \mathfrak{z}(Q,K)(\alpha_c)(\mathfrak{a}'_c(S)).$$

A direct proof is possible (see [128, Proposition 4.13]), but it can also be achieved by an application of Corollary I.27. First, recall from Remark III.2 that the 2-cell α induces a natural transformation

$$\begin{tikzcd} \mathcal{C} \rtimes P \arrow[rr, bend left, "F \rtimes a"] \arrow[rr, bend right, "F' \rtimes a'"'] \arrow[rr, phantom, "\Downarrow\breve{\alpha}"] & & \mathcal{D} \rtimes Q. \end{tikzcd}$$

In turn, $\breve{\alpha}$ induces a 2-cell of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(P,J) \arrow[rr, bend left, "\mathbf{Sh}(F \rtimes a)"] \arrow[rr, bend right, "\mathbf{Sh}(F' \rtimes a')"'] \arrow[rr, phantom, "\Downarrow\mathbf{Sh}(\breve{\alpha})"] & & \mathbf{Sh}(Q,K) \end{tikzcd}$$

by [63, Remark C2.3.5].

Recall also that the Grothendieck topology $K_{\mathfrak{z}(Q,K)}$ on $\mathcal{D} \rtimes Q$ is relatively subcanonical as defined in Definition I.26 (see Remark II.15(ii)). Therefore, by Corollary I.27 the 2-cell of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathfrak{z}(P,J), K_{\mathfrak{z}(P,J)}) \simeq \mathbf{Sh}(P,J) \arrow[rr, bend left, "\mathbf{Sh}(F \rtimes a)"] \arrow[rr, bend right, "\mathbf{Sh}(F' \rtimes a')"'] \arrow[rr, phantom, "\Downarrow\mathbf{Sh}(\breve{\alpha})"] & & \mathbf{Sh}(Q,K) \simeq \mathbf{Sh}(\mathfrak{z}(Q,K), K_{\mathfrak{z}(Q,K)}) \end{tikzcd}$$

induces a natural transformation $F \rtimes \mathfrak{a} \Rightarrow F' \rtimes \mathfrak{a}'$ and hence, since $\mathfrak{z}(P,J)$ has non-empty fibres, also a 2-cell

$$\begin{tikzcd} \mathfrak{z}(P,J) \arrow[rr, bend left, "{(F,\mathfrak{a})}"] \arrow[rr, bend right, "{(F',\mathfrak{a}')}"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathfrak{z}(Q,K). \end{tikzcd}$$

as desired (see Remark III.2).

We now show that, for each doctrinal site $(P, J) \in \mathbf{DocSites}$ and geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$, there is a natural isomorphism of categories

$$\mathbf{DocSites}((P, J), (\mathbb{L}, K_{\mathbb{L}})) \cong \mathbf{GeomDoc}(\mathfrak{Z}(P, J), \mathbb{L}). \tag{IV.iv}$$

The isomorphism on objects is provided by the universal property of the geometric completion. We demonstrate the isomorphism on arrows. Given a pair of morphisms of doctrines $(F, a), (F', a') : (P, J) \rightrightarrows (\mathbb{L}, K_{\mathbb{L}})$ and a natural transformation $\alpha : F \Rightarrow F'$, if $a_c(x) \leqslant \mathbb{L}(\alpha_c)(a'_c(x))$ for all $c \in C$ and $x \in P(c)$, i.e. α defines a 2-cell $\alpha : (F, a) \Rightarrow (F', a')$, then α also defines a 2-cell $\alpha : (F, \mathfrak{a}) \Rightarrow (F', \mathfrak{a}')$ by the 2-functoriality of $\mathfrak{Z}$ shown above. Conversely, if $\mathfrak{a}_c(S) \leqslant \mathbb{L}(\alpha_c)(\mathfrak{a}'_c(S))$ for all $c \in C$ and $S \in \mathfrak{Z}(P, J)(c)$, then

$$a_c(x) = \mathfrak{a}_c(\eta_c^{(P,J)}(x)) \leqslant \mathbb{L}(\alpha_c)(\mathfrak{a}'_c(\eta_c^{(P,J)}(x))) = \mathbb{L}(\alpha_c)(a'_c(x)),$$

and so α defines a 2-cell $\alpha : (F, a) \Rightarrow (F', a')$. Thus, we obtain the isomorphism (IV.iv). Hence, we have demonstrated the strict 2-adjunction

$$\begin{tikzcd} \mathbf{DocSites} \arrow[r, shift left, "\mathfrak{Z}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc} \arrow[l, shift left, hook'] \end{tikzcd}$$

as desired. □

Remark IV.20. The isomorphism (IV.iv) could also be obtained by the more general observation: whenever (Q, K) is a doctrinal site such that each component $\eta_d^{(Q,K)} : Q(d) \to \mathfrak{Z}(Q, K)(d)$ of the unit is injective, for any other doctrinal site (P, J) and a pair of morphisms of doctrinal sites $(F, a), (F, a') : (P, J) \rightrightarrows (Q, K)$, a natural transformation $\alpha : F \Rightarrow F'$ defines a 2-cell $(F, a) \Rightarrow (F', a')$ if and only if α defines a 2-cell $(F, \mathfrak{a}) \Rightarrow (F', \mathfrak{a}')$. This is a consequence of the fact that if $\eta_d^{(Q,K)}$ is injective for each $d \in \mathcal{D}$, then K is a relatively subcanonical topology.

Therefore, the induced functor on hom-categories

$$\mathbf{DocSites}((P, J), (Q, K)) \longrightarrow \mathbf{GeomDoc}(\mathfrak{Z}(P, J), \mathfrak{Z}(Q, K))$$

is full and faithful. The specific isomorphism (IV.iv) can then be obtained by noting that $\eta^{(\mathbb{L}, K_{\mathbb{L}})} : \mathbb{L} \to \mathfrak{Z}(\mathbb{L}, K_{\mathbb{L}})$ is an isomorphism for any geometric doctrine $\mathbb{L}$.

It remains to describe the algebras of the geometric completion monad $(\mathfrak{Z}, \eta, \mu)$ of the adjunction (IV.iii). Since the geometric completion is an idempotent monad, a simple application of [13, Corollary 4.2.4, Volume 2] (extended to the 2-categorical setting) yields the following corollary.

Corollary IV.21. *The algebras for the monad $(\mathfrak{Z}, \eta, \mu)$ coincide with geometric doctrines, i.e.*

$$\mathbf{DocSites}^{\mathfrak{Z}} \simeq \mathbf{GeomDoc}.$$

In particular, by restricting the adjunction (IV.iii), for each category C there is a 2-equivalence

$$(\mathbf{DocSites}/C)^{\mathfrak{Z}} \simeq \mathbf{Loc}\left(\mathbf{Sets}^{C^{\mathrm{op}}}\right)^{\mathrm{op}},$$

where $(\mathbf{DocSites}/C)$ denotes the 1-full 2-subcategory of $\mathbf{DocSites}$ whose objects are doctrinal sites fibred over the category C.

IV.1.4 The geometric completion of a regular site

We are able to combine the geometric completion of a doctrine with the syntactic category construction studied in Section III.3 to define the geometric completion of a regular site, which sends a regular site to a geometric category. Unsurprisingly, this amounts to assigning to each regular site $(\mathcal{C}, K)$ the full subcategory of $\mathbf{Sh}(\mathcal{C}, K)$ spanned by subobjects of representables.

Definition IV.22. We denote by **GeomCat** the 2-category of *geometric categories*, the 2-category

(i) whose objects are geometric categories – regular categories whose subobject lattices have arbitrary joins that are preserved by pullback,

(ii) whose 1-cells are *geometric functors* – regular functors that also preserve joins of subobjects,

(iii) and whose 2-cells are natural transformations between these.

Each geometric category $\mathcal{G}$ can be equipped with the *geometric topology* J_{Geom}, the Grothendieck topology whose covering families are the jointly epimorphic ones, to obtain a regular site $(\mathcal{G}, J_{\mathrm{Geom}})$. In light of Remark III.37, this is the topology whose restriction $J_{\mathrm{Geom}}|_{\mathrm{Sub}_{\mathcal{G}}(g)}$ to the subobject lattice $\mathrm{Sub}_{\mathcal{G}}(g)$, for $g \in \mathcal{G}$, is the topology where

$$\{\, e_i \rightarrowtail h \mid i \in I \,\} \text{ is a } J_{\mathrm{Geom}}|_{\mathrm{Sub}_{\mathcal{G}}(g)}\text{-cover} \iff h = \bigvee_{i \in I} e_i.$$

This assignment of a regular site to a geometric category $\mathcal{G}$ is easily observed to determine a full and faithful 2-embedding $\mathbf{GeomCat} \hookrightarrow \mathbf{RegSites}$.

Theorem IV.23. *There is a pseudo-adjunction*

$$\begin{tikzcd} \mathbf{RegSites} \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomCat} \arrow[l, hook', shift left=2] \end{tikzcd}$$

for which each square in the diagram

$$\begin{tikzcd} \mathbf{ExDocSites} \arrow[r, shift left=2, "\mathfrak{Z}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right=2, "\mathbf{Syn}"'] \arrow[d, phantom, "\dashv"] & \mathbf{GeomDoc}_{\mathrm{cart}} \arrow[l, hook', shift left=2] \arrow[d, shift right=2, "\mathbf{Syn}"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{RegSites} \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right=2, "\mathrm{Sub}_{(-)}"'] & \mathbf{GeomCat} \arrow[l, hook', shift left=2] \arrow[u, shift right=2, "\mathrm{Sub}_{(-)}"'] \end{tikzcd} \tag{IV.v}$$

commutes.

Proof. In order to obtain the commutativity of the diagram (IV.v), we define

$$\mathfrak{Z}^{\mathrm{Cat}} \colon \mathbf{RegSites} \longrightarrow \mathbf{GeomCat}$$

as the composite $\mathbf{Syn} \circ \mathfrak{Z} \circ \mathrm{Sub}_{(-)}$. The pseudo-adjunction is then obtained by the natural equivalences, for each regular site $(C, K) \in \mathbf{RegSites}$ and geometric category $\mathcal{G} \in \mathbf{GeomCat}$,

$$\begin{aligned}\mathbf{GeomCat}(\mathbf{Syn}(\mathfrak{Z}(\mathrm{Sub}_C, K_{\mathrm{Sub}})), \mathcal{G}) &\simeq \mathbf{GeomDoc}_{\mathrm{cart}}(\mathfrak{Z}(\mathrm{Sub}_C, K_{\mathrm{Sub}}), \mathrm{Sub}_{\mathcal{G}}),\\ &\simeq \mathbf{ExDocSites}((\mathrm{Sub}_C, K_{\mathrm{Sub}}), (\mathrm{Sub}_{\mathcal{G}}, K_{\mathrm{Sub}_{\mathcal{G}}})),\\ &\simeq \mathbf{RegSites}((C, K), (\mathcal{G}, J_{\mathrm{Geom}})),\end{aligned}$$

where in the last equivalence we have used that $\mathrm{Sub}_{(-)} \colon \mathbf{RegSites} \to \mathbf{ExDocSites}$ is full and faithful. □

IV.2 Coarse geometric completions

Because the geometric completion takes a Grothendieck topology as a second argument, it is an idempotent completion (see Theorem IV.16). This is in contrast to many of the other completions of doctrines considered in the literature (e.g. Trotta's existential completion [119]). The geometric completion would not be idempotent if we did not have the ability to take suitable topologies as a second argument.

Consider the terminal frame $\mathbf{2}$. Being a frame, there is a canonical isomorphism J_{can}-$\mathbf{Idl}(\mathbf{2}) \cong \mathbf{2}$, but one can easily calculate that J_{triv}-$\mathbf{Idl}(\mathbf{2})$ is the 3-element frame $\mathbf{3}$ (i.e. the opens of the Sierpinski space). We can interpret this behaviour as a 'loss of information' by taking a *coarser* Grothendieck topology $J_{\mathrm{triv}} \subseteq J_{\mathrm{can}}$ on $\mathbf{2}$. In order to relate the geometric completion to other completions of doctrines considered in the literature, we consider in this section the behaviour of the geometric completion for doctrines when, for each geometric doctrine $\mathbb{L}$, we deliberately choose a coarser Grothendieck topology $J^A_{\mathbb{L}} \subseteq K_{\mathbb{L}}$ on the category $C \rtimes \mathbb{L}$ (or indeed forget the Grothendieck topology entirely by assigning the trivial topology J_{triv} to $C \rtimes \mathbb{L}$).

We thus arrive at the notion of a *coarse geometric completion* – a 2-monad $\mathfrak{Z}_A$ acting on a 2-full 2-subcategory of $\mathbf{DocSites}$. As evidenced by the example given above, this monad $\mathfrak{Z}_A$ is no longer idempotent (unless each $J^A_{\mathbb{L}}$ is chosen to be $K_{\mathbb{L}}$), unlike the geometric completion monad $\mathfrak{Z}$. We will observe that each coarse geometric completion is instead *lax-idempotent*. After establishing the lax-idempotency of a coarse geometric completion in Corollary IV.28, we demonstrate in Corollary IV.30 how this yields a lax-idempotent geometric completion for cartesian, regular and coherent categories.

Definition IV.24. A *coarse geometric completion* consists of the following data.

(i) We are given a 2-full 2-subcategory A-$\mathbf{Doc} \subseteq \mathbf{DocSites}$. The objects of A-$\mathbf{Doc}$ we call A-doctrines and their morphisms we call A-doctrine morphisms.

(ii) There is a 2-subcategory $\mathbf{GeomDoc}_A \subseteq \mathbf{GeomDoc}$ which is full on 1-cells and 2-cells satisfying the following conditions.

   a) For each $\mathbb{L} \in \mathbf{GeomDoc}_A$, there is a choice of Grothendieck topology $J^A_{\mathbb{L}}$ on the category $C \rtimes \mathbb{L}$ which is coarser than the topology $K_{\mathbb{L}}$, i.e. $J^A_{\mathbb{L}} \subseteq K_{\mathbb{L}}$, such that $(\mathbb{L}, J^A_{\mathbb{L}})$ is an object of A-$\mathbf{Doc}$. Moreover, the choice of topology $J^A_{\mathbb{L}}$ is functorial in the sense that, for each morphism of geometric doctrines

$(F, a)\colon \mathbb{L} \to \mathbb{L}'$, there is a morphism of A-doctrines

$$(F, a)\colon (\mathbb{L}, J^A_{\mathbb{L}}) \longrightarrow (\mathbb{L}', J^A_{\mathbb{L}'}).$$

In other words, there is a 2-full 2-embedding $\mathbf{GeomDoc}_A \hookrightarrow A\text{-}\mathbf{Doc}$.

b) For each object $(P, J) \in A\text{-}\mathbf{Doc}$, the geometric completion $\mathfrak{z}(P, J)$ is contained in $\mathbf{GeomDoc}_A$ and the unit $\eta^{(P,J)}\colon P \to \mathfrak{z}(P, J)$ defines a morphism of A-doctrines

$$\eta^{(P,J)}\colon (P, J) \longrightarrow (\mathfrak{z}(P, J), J^A_{\mathfrak{z}(P,J)}).$$

Theorem IV.25. *Let $A\text{-}\mathbf{Doc} \subseteq \mathbf{DocSites}$ and $\mathbf{GeomDoc}_A \subseteq \mathbf{GeomDoc}$ define a coarse geometric completion. There is a strict 2-adjunction*

$$\begin{tikzcd} A\text{-}\mathbf{Doc} \arrow[r, shift left, "\mathfrak{z}_A"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_A \arrow[l, hook', shift left] \end{tikzcd}$$

where $\mathfrak{z}_A$ is the 2-functor

$$A\text{-}\mathbf{Doc} \hookrightarrow \mathbf{DocSites} \xrightarrow{\mathfrak{z}} \mathbf{GeomDoc}.$$

Proof. For each $(P, J) \in A\text{-}\mathbf{Doc}$ and $\mathbb{L} \in \mathbf{GeomDoc}_A$, the natural isomorphism on objects of the categories

$$A\text{-}\mathbf{Doc}((P, J), (\mathbb{L}, J^A_{\mathbb{L}})) \cong \mathbf{GeomDoc}_A(\mathfrak{z}_A(P, J), \mathbb{L}) \tag{IV.vi}$$

acts by sending a morphism of geometric doctrines $(F, a)\colon \mathfrak{z}_A(P, J) \to \mathbb{L}$ to the composite

$$(P, J) \xrightarrow{\eta^{(P,J)}} (\mathfrak{z}_A(P, J), J^A_{\mathfrak{z}_A(P,J)}) \xrightarrow{(F,a)} (\mathbb{L}, J^A_{\mathbb{L}})$$

and, vice versa, sending an arrow $(F, a)\colon (P, J) \to (\mathbb{L}, J^A_{\mathbb{L}})$ to the morphism of geometric doctrines $(F, \mathfrak{a})$ as induced by the diagram

$$\begin{tikzcd} (P, J) \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "{(F,a)}"'] & (\mathfrak{z}(P, J), J^A_{\mathfrak{z}(P,J)}) \arrow[r] \arrow[d, dashed, "{(F,\mathfrak{a})}"] & (\mathfrak{z}(P, J), K_{\mathfrak{z}(P,J)}) \arrow[d, dashed, "{(F,\mathfrak{a})}"] \\ & (\mathbb{L}, J^A_{\mathbb{L}}) \arrow[r] & (\mathbb{L}, K_{\mathbb{L}}) \end{tikzcd}$$

and Theorem IV.16. That this extends to an isomorphism on arrows, and hence the isomorphism of categories (IV.vi), follows from Proposition IV.19 and the fact that $A\text{-}\mathbf{Doc} \subseteq \mathbf{DocSites}$ and $\mathbf{GeomDoc}_A \subseteq \mathbf{GeomDoc}$ are both full on 2-cells. □

Of course, $\mathbf{2}$ is a quotient frame (or *sublocale*) of $\mathbf{3}$. Similarly, the coarse geometric completion of a geometric doctrine $\mathfrak{z}(\mathbb{L}, J^A_{\mathbb{L}})$ is related to the geometric doctrine $\mathbb{L}$ by a pointwise surjective morphism of geometric doctrines $\mathfrak{z}(\mathbb{L}, J^A_{\mathbb{L}}) \to \mathbb{L}$ (or internal sublocale embedding) corresponding to the inclusion of the subtopos $\mathbf{Sh}(C \rtimes \mathbb{L}, K_{\mathbb{L}}) \rightarrowtail \mathbf{Sh}(C \rtimes \mathbb{L}, J^A_{\mathbb{L}})$ (see Proposition II.28 – if $J^A_{\mathbb{L}}$ is the trivial topology, the morphism $\mathfrak{z}(\mathbb{L}, J_{\mathrm{triv}}) \to \mathbb{L}$ is precisely the $K_{\mathbb{L}}$-closure operation from Definition IV.5).

Lax idempotency for coarse geometric completions. As previously mentioned, the strict 2-adjunction

$$\begin{tikzcd} A\text{-}\mathbf{Doc} \arrow[r, shift left, "\mathfrak{Z}_A"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_A \arrow[l, shift left, hook] \end{tikzcd}$$

of a coarse geometric completion is not necessarily idempotent. We dedicate the remainder of this section to showing that $\mathfrak{Z}_A$ satisfies a weaker form of idempotency: *lax-idempotency*.

Let $T\colon \mathcal{C} \to \mathcal{C}$ be a 2-monad with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to T$ and multiplication $\mu\colon T^2 \to T$. The 2-monad T is *lax-idempotent*[1] if the composites of the diagram

$$\begin{tikzcd} TA \arrow[r, shift right, "\eta_{TA}"'] & T^2A, \arrow[l, shift right, "\mu_A"'] \end{tikzcd}$$

although perhaps not strictly equal to the identities id_{TA} and id_{T^2A}, as is the case for an idempotent monad, can instead be related by canonical 2-cells such that there is an adjunction $\mu_A \dashv \eta_{TA}$ (see [74, Proposition 1.2]). Specifically, we require a require a 2-cell

$$\begin{tikzcd} TA \arrow[r, bend left=50, "T\eta_A"] \arrow[r, bend right=50, "\eta_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A, \end{tikzcd}$$

natural in A, such that the horizontal composites

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & TA \arrow[r, bend left=50, "T\eta_A"] \arrow[r, bend right=50, "\eta_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A, \end{tikzcd} \quad \begin{tikzcd} TA \arrow[r, bend left=50, "T\eta_A"] \arrow[r, bend right=50, "\eta_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A \arrow[r, "\mu_A"] & A \end{tikzcd}$$

are both identity 2-cells (see [74, Definition 1.1]).

Often it can be more tractable, if circumlocutory, to demonstrate lax-idempotency by an equivalent condition regarding the algebras of the monad. Recall from [72] that, given a pair of (strict) T-algebras (A, a) and (B, a), a *lax morphism* of T-algebras is a pair (f, α) where $f\colon A \to B$ is an arrow of $\mathcal{C}$ while α is a 2-cell $\alpha\colon b \circ Tf \Rightarrow f \circ a$ that fills the square

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

[1] In [74], lax-idempotent monads are called *KZ-doctrines*. However, using this terminology would be confusing in the context of doctrines in the sense of Lawvere.

and satisfies the coherence conditions

$$\begin{tikzcd} T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] & T^2B \arrow[d, "\mu_B"] \\ TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} = \begin{tikzcd} T^2A \arrow[r, "T^2f"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\Downarrow T\alpha"] & T^2B \arrow[d, "TB"] \\ TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{IV.vii}$$

and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[dd, "\mathrm{id}_A"'] & B \arrow[dd, "\mathrm{id}_B"] \\ & \\ A \arrow[r, "f"] & B. \end{tikzcd} \tag{IV.viii}$$

It is shown in [72, Theorem 6.2] that T is lax-idempotent if and only if for each pair (A, a) and (B, b) of (strict) T-algebras and a morphism $f : A \to B$ there is a unique 2-cell $\alpha : b \circ Tf \Rightarrow f \circ a$ such that $(f, \alpha): (A, a) \to (B, b)$ is a lax morphism of T-algebras.

We require two lemmas concerning the algebras of $\mathfrak{Z}_A$ in order to demonstrate that a coarse geometric completion $\mathfrak{Z}_A : A\text{-}\mathbf{Doc} \to A\text{-}\mathbf{Doc}$ is lax-idempotent. Since, in the pertinent examples of coarse geometric completions we will consider in Examples IV.29, the Grothendieck topology J given on $C \rtimes P$ for an A-doctrine $(P, J) \in A\text{-}\mathbf{Doc}$ is chosen for us, in what follows we simplify notation and denote the object (P, J) of $A\text{-}\mathbf{Doc}$ by simply P. Also in aid of legibility, if $(G, b) = \xi : P \to Q$ is a morphism of A-doctrines, we will abuse notation and write ξ for the natural transformation $b : P \Rightarrow Q \circ G^{\mathrm{op}}$ as well.

Lemma IV.26. *Let $P : C^{\mathrm{op}} \to$ **PreOrd** and $Q : \mathcal{D}^{\mathrm{op}} \to$ **PreOrd** be a pair of A-doctrines and let $\xi : \mathfrak{Z}_A(P) \Rightarrow P$ and $\zeta : \mathfrak{Z}_A(Q) \Rightarrow Q$ be natural transformations such that the triangles*

$$\begin{tikzcd} P \arrow[r, "\eta^P"] \arrow[dr, equal] & \mathfrak{Z}_A(P) \arrow[d, "\xi"] \\ & P, \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\eta^Q"] \arrow[dr, equal] & \mathfrak{Z}_A(Q) \arrow[d, "\zeta"] \\ & Q \end{tikzcd}$$

commute. Given a morphism of doctrine $(F, a): P \to Q$, for each arrow $d \xrightarrow{f} c \in C$ and $x \in P(d)$, there is an inequality

$$\zeta_c \circ \exists_{\mathfrak{Z}_A(Q)(F(f))} \circ \eta^Q_d \circ a_d(x) \leqslant a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x).$$

Proof. Firstly, using the inequality $\eta^P_d(x) \leqslant \mathfrak{Z}_A(P)(f) \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x)$, we deduce that

$$\begin{aligned} \eta^Q_d \circ a_d(x) &= \eta^Q_d \circ a_d \circ \xi_d \circ \eta^P_d(x), \\ &\leqslant \eta^Q_d \circ a_d \circ \xi_d \circ \mathfrak{Z}_A(P)(f) \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x), \\ &= \mathfrak{Z}_A(Q)(F(f)) \circ \eta^Q_c \circ a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x). \end{aligned}$$

Thus, by the adjunction $\exists_{\mathfrak{Z}(Q)(F(f))} \dashv \mathfrak{Z}(Q)(F(f))$, we have that

$$\exists_{\mathfrak{Z}_A(Q)(F(f))} \circ \eta_d^Q \circ a_d(x) \leqslant \eta_c^Q \circ a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta_d^P(x),$$

and we therefore obtain the desired inequality

$$\begin{aligned}\zeta_c \circ \exists_{\mathfrak{Z}_A(Q)(F(f))} \circ \eta_d^Q \circ a_d(x) &\leqslant \zeta_c \circ \eta_c^Q \circ a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta_d^P(x), \\ &= a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta_d^P(x).\end{aligned}$$

□

Lemma IV.27. *If (P, ξ) is a $\mathfrak{Z}_A$-algebra, then $\xi_c \colon \mathfrak{Z}_A(P)(c) \to P(c)$ preserves joins for all $c \in \mathcal{C}$.*

Proof. Let us first show that $P(c)$ must have all joins. For a subset $\{ x_i \mid i \in I \}$ of $P(c)$, we claim that the join $\bigvee_{i\in I} x_i$ is given by $\xi_c\left(\bigvee_{i\in I} \eta_c^P(x)\right)$. For each $i \in I$, we have that

$$x_i = \xi_c \circ \eta_c^P(x) \leqslant \xi_c\left(\bigvee_{i\in I} \eta_c^P(x)\right)$$

while, if given $y \in P(c)$ with $x_i \leqslant y$ for all $i \in I$, we have the converse inequality

$$\xi_c\left(\bigvee_{i\in I} \eta_c^P(x)\right) \leqslant \xi_c \circ \eta_c^P(y) = y.$$

Hence, the join $\bigvee_{i\in I} x_i$ is given by $\xi_c\left(\bigvee_{i\in I} \eta_c^P(x)\right)$.

To show that ξ_c preserves these joins, we first observe that the diagrams

$$\begin{tikzcd} & \mathfrak{Z}_A\mathfrak{Z}_A(P) \arrow[r, "\mathfrak{Z}_A(\xi)"] \arrow[d, "\mu^P"] & \mathfrak{Z}_A(P) \arrow[d, "\xi"] \\ \mathfrak{Z}_A(P) \arrow[ur, "\eta^{\mathfrak{Z}_A(P)}"] \arrow[r, equal] & \mathfrak{Z}_A(P) \arrow[r, "\xi"] & P, \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Z}_A(P) \arrow[r, "\xi"] \arrow[d, "\eta^{\mathfrak{Z}_A(P)}"'] & P \arrow[d, "\eta^P"] \\ \mathfrak{Z}_A\mathfrak{Z}_A(P) \arrow[r, "\mathfrak{Z}_A(\xi)"] & \mathfrak{Z}_A(P) \end{tikzcd}$$

both commute – the left-hand diagram commutes since $\mathfrak{Z}_A$ is a monad and (P, ξ) is a $\mathfrak{Z}_A$-algebra, while the right-hand square commutes as η is natural. Note also that μ^P and $\mathfrak{Z}_A(\xi)$ are morphisms of geometric doctrines. In particular, for each $c \in \mathcal{C}$, both μ_c^P and $\mathfrak{Z}_A(\xi)_c$ commute with all joins.

Therefore, given a subset $\{ S_i \mid i \in I \} \subseteq \mathfrak{Z}_A(P)(c)$, we observe that

$$\begin{aligned}\bigvee_{i\in I} \xi_c(S_i) &= \xi_c\left(\bigvee_{i\in I} \eta_c^P \circ \xi_c(S_i)\right), \\ &= \xi_c\left(\bigvee_{i\in I} \mathfrak{Z}_A(\xi)_c \circ \eta_c^{\mathfrak{Z}_A(P)}(S_i)\right), \\ &= \xi_c \circ \mathfrak{Z}_A(\xi)_c\left(\bigvee_{i\in I} \eta_c^{\mathfrak{Z}_A(P)}(S_i)\right), \\ &= \xi_c \circ \mu_c^P\left(\bigvee_{i\in I} \eta_c^{\mathfrak{Z}_A(P)}(S_i)\right), \\ &= \xi_c\left(\bigvee_{i\in I} \mu_c^P \circ \eta_c^{\mathfrak{Z}_A(P)}(S_i)\right) = \xi_c\left(\bigvee_{i\in I} S_i\right)\end{aligned}$$

and hence joins are indeed preserved. □

Finally we complete the proof that $\mathfrak{Z}_A$ is lax-idempotent. The argument is reminiscent of that found in [119, Theorem 5.6].

Corollary IV.28. *Each coarse geometric completion $\mathfrak{Z}_A \colon A\text{-}\mathbf{Doc} \to A\text{-}\mathbf{Doc}$ is lax-idempotent.*

Proof. Let (P, ξ) and (Q, ζ) be algebras of the 2-monad $\mathfrak{Z}_A$. For each morphism of A-doctrines (F, a), we first demonstrate that the identity transformation $\mathrm{id}_F \colon F \Rightarrow F$ defines a 2-cell that fills the square

$$\begin{tikzcd} \mathfrak{Z}_A(P) \arrow[r, "{(F,\mathfrak{a})}"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_F"] & \mathfrak{Z}_A(Q) \arrow[d, "\zeta"] \\ P \arrow[r, "{(F,a)}"'] & Q. \end{tikzcd}$$

We thus need to demonstrate, for all $S \in \mathfrak{Z}_A(P)(c)$, the inequality

$$\zeta_c \circ \mathfrak{a}_c(S) \leqslant a_c \circ \xi_c(S).$$

By combining Remark IV.17, Lemma IV.26 and Lemma IV.27, we obtain the desired inequality:

$$\begin{aligned} \zeta_c \circ \mathfrak{a}_c(S) &= \zeta_c \left(\bigvee_{(f,x) \in S} \exists_{\mathfrak{Z}_A(Q)(F(f))} \circ \eta^Q_d \circ a_d(x) \right), \\ &= \bigvee_{(f,x) \in S} \zeta_c \circ \exists_{\mathfrak{Z}_A(Q)(F(f))} \circ \eta^Q_d \circ a_d(x), \\ &\leqslant \bigvee_{(f,x) \in S} a_c \circ \xi_c \circ \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x), \\ &\leqslant a_c \circ \xi_c \left(\bigvee_{(f,x) \in S} \exists_{\mathfrak{Z}_A(P)(f)} \circ \eta^P_d(x) \right) = a_c \circ \xi_c(S). \end{aligned}$$

Its trivially shown that $((F, a), \mathrm{id}_F)$ satisfies the coherence conditions (IV.vii) and (IV.viii).

For any other 2-cell $\alpha \colon \zeta \circ (F, \mathfrak{a}) \Rightarrow (F, a) \circ \xi$ satisfying the coherence condition

$$\begin{tikzcd} P \arrow[r, "{(F,a)}"] \arrow[d, "\eta^P"'] & Q \arrow[d, "\eta^Q"] \\ \mathfrak{Z}_A(P) \arrow[r, "{(F,\mathfrak{a})}"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\Downarrow \alpha"] & \mathfrak{Z}_A(Q) \arrow[d, "\zeta"] \\ A \arrow[r, "{(F,a)}"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} P \arrow[r, "{(F,a)}"] \arrow[d, "\mathrm{id}_P"'] & Q \arrow[d, "\mathrm{id}_Q"] \\ P \arrow[r, "{(F,a)}"] & Q, \end{tikzcd}$$

the equality $\alpha = \mathrm{id}_F$ is forced, and so $((F, a), \mathrm{id}_F)$ is the unique such lax $\mathfrak{Z}_A$-algebra morphism. ◻

Examples IV.29. We obtain the following lax-idempotent 2-monads as applications of Theorem IV.25 and Corollary IV.28.

(i) By $\mathbf{Doc}_{\mathrm{flat}}$ denote the 1-full 2-subcategory of **DocSites** on objects of the form (P, J_{triv}). Equivalently, $\mathbf{Doc}_{\mathrm{flat}}$ is the 2-full 2-subcategory of **Doc** on doctrines and *flat* morphisms of doctrines. The assignment of the trivial topology J_{triv} to each geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$ yields a 2-embedding $\mathbf{GeomDoc} \hookrightarrow \mathbf{Doc}$ that satisfies the conditions of Definition IV.24. Thus, we obtain a coarse geometric completion that we will call the *free* geometric completion

$$\begin{tikzcd} \mathbf{Doc}_{\mathrm{flat}} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Fr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}. \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

In particular, this restricts to a strict 2-adjunction

$$\begin{tikzcd} \mathbf{PrimDoc} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Fr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_{\mathrm{cart}}, \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

between the 2-category of primary doctrines and the 2-category of geometric doctrines indexed over cartesian base categories. The free geometric completion coincides with the completion studied in [38, §3.1.3].

(ii) There is a 2-embedding of **GeomDoc** into the 2-full 2-subcategory **RelExDoc** of **DocSites** of relative existential doctrines, given by sending a geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$ to $(\mathbb{L}, J_{\mathrm{Ex}}) \in \mathbf{RelExDoc}$, satisfying the conditions of Definition IV.24. Hence, we obtain a coarse geometric completion that we call the *existential* geometric completion

$$\begin{tikzcd} \mathbf{RelExDoc} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Ex}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}, \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

This strict 2-adjunction restricts to the 2-subcategories of existential doctrines and geometric doctrines over a cartesian base category

$$\begin{tikzcd} \mathbf{ExDoc} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Ex}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_{\mathrm{cart}}. \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

By Proposition IV.6, the existential geometric completion is a pointwise construction.

(iii) Similarly, we obtain a coarse coherent completion for relative coherent doctrines, the *coherent* geometric completion

$$\begin{tikzcd} \mathbf{RelCohDoc} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Coh}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}, \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

where $\mathbf{GeomDoc} \hookrightarrow \mathbf{RelCohDoc}$ is the 2-embedding that sends a geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$ to $(\mathbb{L}, J_{\mathrm{Coh}}) \in \mathbf{RelCohDoc}$. Once again, this restricts to a strict 2-adjunction

$$\begin{tikzcd} \mathbf{CohDoc} \arrow[r, shift left=1.5ex, "\mathfrak{z}_{\mathrm{Coh}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_{\mathrm{cart}}. \arrow[l, hook', shift left=1.5ex] \end{tikzcd}$$

Once again, by Proposition IV.6, this is a pointwise construction.

Coarse geometric completions for categories. We now relate how the coarse geometric completions considered in Examples IV.29 interact with the syntactic category construction from Section III.3. We will obtain (coarse) geometric completions for cartesian categories, regular categories and coherent categories.

A *coherent category* (see [63, §A1.4], also called a *logical category* in [87]) is a regular category whose subobject lattices $\mathrm{Sub}_{\mathcal{C}}(c)$ have finite joins and, for each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, $\mathrm{Sub}_{\mathcal{C}}(f)$ preserves these finite joins. A *coherent functor* $F\colon \mathcal{C} \to \mathcal{D}$, between coherent categories, is a regular functor that preserves finite joins as well. We denote by **Coh** the 2-category of coherent categories, coherent functors and natural transformations between these.

The 2-functors $\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Ex}}\colon \mathbf{Reg} \to \mathbf{GeomCat}$ and $\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Coh}}\colon \mathbf{Coh} \to \mathbf{GeomCat}$ constructed below in Corollary IV.30 are evidently given by the composites

$$\mathbf{Reg} \hookrightarrow \mathbf{RegSites} \xrightarrow{\mathfrak{Z}^{\mathrm{Cat}}} \mathbf{GeomCat}$$

and

$$\mathbf{Coh} \hookrightarrow \mathbf{RegSites} \xrightarrow{\mathfrak{Z}^{\mathrm{Cat}}} \mathbf{GeomCat},$$

where $\mathbf{Reg} \hookrightarrow \mathbf{RegSites}$ (respectively, $\mathbf{Coh} \hookrightarrow \mathbf{RegSites}$) denotes the 2-embedding that sends a regular (resp., coherent) category $\mathcal{C}$ to the regular site $(\mathcal{C}, J_{\mathrm{Reg}})$ (resp., $(\mathcal{C}, J_{\mathrm{Coh}})$). Here J_{Reg} denotes the *regular topology* and J_{Coh} denotes the *coherent topology* (see [63, Examples A2.1.11]).

Corollary IV.30. *There are lax-idempotent pseudo-adjunctions:*

$$\text{(i)}\quad \begin{tikzcd} \mathbf{Cart} \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Fr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomCat}, \arrow[l, hook', shift left=2] \end{tikzcd}$$

$$\text{(ii)}\quad \begin{tikzcd} \mathbf{Reg} \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Ex}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomCat}, \arrow[l, hook', shift left=2] \end{tikzcd}$$

$$\text{(iii)}\quad \begin{tikzcd} \mathbf{Coh} \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Coh}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomCat}. \arrow[l, hook', shift left=2] \end{tikzcd}$$

Proof. We will only spell out the proof for (i), the other pseudo-adjoints being constructed in a similar fashion. We define $\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Fr}}\colon \mathbf{Cart} \to \mathbf{GeomCat}$ as the composite $\mathbf{Syn} \circ \mathfrak{Z}_{\mathrm{Fr}} \circ \mathrm{Sub}_{(-)}$, as in the diagram

$$\begin{tikzcd} \mathbf{PrimDoc} \arrow[r, shift left=2, "\mathfrak{Z}_{\mathrm{Fr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomDoc}_{\mathrm{cart}} \arrow[l, hook', shift left=2] \arrow[d, shift right=2, "\mathbf{Syn}"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Cart} \arrow[u, "\mathrm{Sub}_{(-)}"] \arrow[r, shift left=2, "\mathfrak{Z}^{\mathrm{Cat}}_{\mathrm{Fr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GeomCat}. \arrow[l, hook', shift left=2] \arrow[u, shift right=2, "\mathrm{Sub}_{(-)}"'] \end{tikzcd}$$

The required natural equivalence of categories, for each $\mathcal{C} \in \mathbf{Cart}$ and $\mathcal{G} \in \mathbf{GeomCat}$,

$$\mathbf{Cart}(\mathcal{C}, \mathcal{G}) \simeq \mathbf{GeomCat}(\mathbf{Syn}(\mathfrak{Z}_{\mathrm{Fr}}(\mathrm{Sub}_{\mathcal{C}})), \mathcal{G})$$

follows by the chain of equivalences

$$\begin{aligned}\mathbf{Cart}(\mathcal{C}, \mathcal{G}) &\simeq \mathbf{PrimDoc}(\mathrm{Sub}_{\mathcal{C}}, \mathrm{Sub}_{\mathcal{G}}),\\ &\simeq \mathbf{GeomDoc}_{\mathrm{cart}}(\mathfrak{Z}_{\mathrm{Fr}}(\mathrm{Sub}_{\mathcal{C}}), \mathrm{Sub}_{\mathcal{G}}),\\ &\simeq \mathbf{GeomCat}(\mathbf{Syn}(\mathfrak{Z}_{\mathrm{Fr}}(\mathrm{Sub}_{\mathcal{C}})), \mathcal{G}),\end{aligned}$$

where we have used that $\mathrm{Sub}_{(-)}$ is full and faithful. □

IV.3 Subgeometric completions

Having hinted at the existence of subgeometric completions throughout, we finally turn to their systematic treatment. The term subgeometric completion is intended to convey the following vague sense: a completion TP of a doctrine P is 'subgeometric' if the data added by T can be 'seen' by a certain Grothendieck topology J^T on the category $\mathcal{C} \rtimes TP$, and has the property that $\mathfrak{Z}(P, J_{\mathrm{triv}}) \cong \mathfrak{Z}(TP, J^T)$ – i.e. freely geometrically completing P is the same as completing P according to T, keeping track of this new information by J^T, and then geometrically completing. We have already observed this phenomenon in Section IV.1.1 with the free top completion, and we will see further examples below. It is this vague notion of 'subgeometricity' that we seek to formalise in this section.

We proceed as follows.

- Immediately below in Section IV.3.1 we present another motivating example for the theory of subgeometric completions: we demonstrate that the existential completion of a primary doctrine due to Trotta [119] satisfies our vague understanding of subgeometricity as stated above.
- We use this, and our study of the free top completion in Section IV.1.1, as intuition when introducing the formal definition of a subgeometric completion in Section IV.3.2. We also discuss sufficient conditions under which a subgeometric completion can automatically be deduced to be lax-idempotent.
- In the remaining two subsections, Section IV.3.3 and Section IV.3.4, we discuss several examples of subgeometric completions. In the former, we discuss subgeometric completions obtained by considering special 'compatible' subdoctrines of the free geometric completion. In this way, we recover the existential completion as well as the coherent completion for primary doctrines. We also relate these completions of primary doctrines to the corresponding *regular completion* and *coherent completion* of cartesian categories (see [27]). Finally, in the latter subsection, we give examples of 'pointwise' subgeometric completions.

IV.3.1 The existential completion is subgeometric

We begin by explicitly describing the free geometric completion $\mathfrak{Z}_{\mathrm{Fr}}(P)$ of a primary doctrine $P \in \mathbf{PrimDoc}$ as defined in Examples IV.29(i). This is the geometric doctrine $\mathfrak{Z}(P, J_{\mathrm{triv}}) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ and thus, by Construction IV.4, can be described in the following way.

(i) For each object c of $\mathcal{C}$, an element S of $\mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ is a set of pairs (f, x), where $d \xrightarrow{f} c$ is an arrow of $\mathcal{C}$ and $x \in P(d)$, such that if $(f, x) \in S$, for each arrow $e \xrightarrow{g} d$ of $\mathcal{C}$ and $y \in P(e)$, if $y \leqslant P(g)(x)$ then $(f \circ g, y) \in S$ too. We order $\mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ by inclusion.

(ii) For each arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, $\mathfrak{Z}_{\mathrm{Fr}}(P)(f) \colon \mathfrak{Z}_{\mathrm{Fr}}(P)(c) \to \mathfrak{Z}_{\mathrm{Fr}}(P)(d)$ sends $S \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ to

$$f^*(S) = \{ (g, y) \,\big|\, (f \circ g, y) \in S \} \in \mathfrak{Z}_{\mathrm{Fr}}(P)(d).$$

The description of the free geometric completion $\mathfrak{Z}_{\mathrm{Fr}}(P)$ given above is markedly similar to the *existential completion* of a primary doctrine established in [119, §4], which we recall below. We will be able to relate the two: the free geometric completion of a primary doctrine can be computed as the existential completion followed by the existential geometric completion – i.e. the pointwise free join completion (see Examples IV.29(ii)).

The existential completion. Recall from [119] that the existential completion of a primary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ is the functor $P^{\exists} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ defined as follows.

(i) Let c be an object of $\mathcal{C}$. Consider the set whose elements are pairs (f, x) where $d \xrightarrow{f} c$ is an arrow of $\mathcal{C}$ and $x \in P(d)$. We order this set by setting $(g, y) \leqslant (f, x)$ if there is an arrow $e \xrightarrow{h} d$, making the triangle commute

$$\begin{tikzcd} e \arrow[d, "h"'] \arrow[dr, "g"] & \\ d \arrow[r, "f"] & c, \end{tikzcd}$$

such that $y \leqslant P(h)(x)$. We define $P^{\exists}(c)$ as the poset obtained when we identify two elements such that $(f, x) \leqslant (g, y)$ and $(g, y) \leqslant (f, x)$. Just as in [119], we will abuse notation and not differentiate between the pair (f, x) and its equivalence class.

(ii) Given an arrow $e \xrightarrow{g} c$ of $\mathcal{C}$, the map $P^{\exists}(g) \colon P^{\exists}(c) \to P^{\exists}(e)$ acts by sending an element $(f, x) \in P^{\exists}(c)$ to $(k, P(h)(x)) \in P^{\exists}(e)$, where

$$\begin{tikzcd} e \times_c d \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, "g"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

This is the 'existential completion' of P in following sense.

(i) For each arrow $e \xrightarrow{g} c$ of $\mathcal{C}$, the map $P^{\exists}(g) \colon P^{\exists}(c) \to P^{\exists}(e)$ has a left adjoint $\exists_{P^{\exists}(g)}$ that sends $(f, x) \in P^{\exists}(e)$ to $(g \circ f, x) \in P^{\exists}(d)$. With these left adjoints, the doctrine $P^{\exists}$ satisfies the Frobenius and Beck-Chevalley conditions (see [119, Proposition 4.2 & Theorem 4.3]).

(ii) There is a natural transformation $\iota_P \colon P \to P^{\exists}$ given by sending $x \in P(c)$ to $(\mathrm{id}_c, x) \in P^{\exists}(c)$ (see [119, Proposition 4.10]).

(iii) Given an existential doctrine $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{MSLat}$, for each morphism of primary doctrines $(F, a)\colon P \to Q$, there is a unique natural transformation $\alpha^{\exists}\colon P^{\exists} \Rightarrow Q\circ F^{\mathrm{op}}$ such that:

a) the triangle

$$\begin{tikzcd} P \arrow[r, "\iota_P"] \arrow[dr, "\alpha"'] & P^{\exists} \arrow[d, "\alpha^{\exists}"] \\ & Q \circ F^{\mathrm{op}} \end{tikzcd}$$

commutes,

b) for each arrow $e \xrightarrow{g} c$ of $\mathcal{C}$, the square

$$\begin{tikzcd} P^{\exists}(c) \arrow[d, "{\alpha^{\exists}}_c"'] & P^{\exists}(e) \arrow[l, "\exists_{P^{\exists}(g)}"'] \arrow[d, "{\alpha^{\exists}}_e"] \\ Q(F(c)) & Q(F(e)) \arrow[l, "\exists_{Q(F(g))}"'] \end{tikzcd}$$

commutes (see [119, Theorem 4.14]).

In [119, Proposition 4.9], it is shown that the existential completion defines a 2-functor

$$(-)^{\exists}\colon \mathbf{PrimDoc} \longrightarrow \mathbf{ExDoc}.$$

We can now observe that the existential completion satisfies our loose notion of 'subgeometricity'.

Proposition IV.31. *For each primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$, there is a natural isomorphism*

$$\mathfrak{Z}_{\mathrm{Fr}}(P) \cong \mathfrak{Z}_{\mathrm{Ex}}(P^{\exists}) = \mathfrak{Z}(P^{\exists}, J_{\mathrm{Ex}}).$$

Proof. This is immediate since the data of a down-set of $P^{\exists}(c)$, i.e. an element of $\mathfrak{Z}_{\mathrm{Ex}}(P^{\exists})$ by Proposition IV.6, is precisely the data of an element $S \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$. □

Remark IV.32. Given a primary doctrine P, the construction presented above of its existential completion $P^{\exists}$ is slightly simplified to that found in [119]. Namely, we have added a left adjoint $\exists_{P^{\exists}(g)}$ to $P^{\exists}(g)$ for every arrow $e \xrightarrow{g} c$ of $\mathcal{C}$, whereas in [119] a generalised construction is given that freely adds a left adjoint $\exists_{P^{\exists}(g)}$ to $P^{\exists}(g)$ for arrows in a chosen class Λ of morphisms of $\mathcal{C}$ closed under pullbacks and compositions and containing all identities.

It is not hard to generalise our exposition to show that this modified existential completion is also subgeometric. In Proposition IV.31, the Grothendieck topology J_{Ex} is replaced by the topology $J_{(\mathrm{Ex},\Lambda)}$, whose covering sieves are precisely those generated by the singleton arrows

$$(d, x) \xrightarrow{f} (c, \exists_{P^{\exists}(f)}x),$$

for each arrow $d \xrightarrow{f} c \in \Lambda$. The conditions on Λ are precisely what are needed to ensure that $J_{(\mathrm{Ex},\Lambda)}$ satisfies definition of a Grothendieck topology – e.g., pullback stability corresponds to the stability condition on $J_{(\mathrm{Ex},\Lambda)}$. As follows from Examples III.10(ii), taking Λ as the class of product projections corresponds to freely completing with respect to existential quantification, while taking Λ as the class of diagonals corresponds to freely completing with respect to an equality predicate.

IV.3.2 Generalised subgeometric completions

We now develop an abstract framework which captures the notion of a *subgeometric completion*. We also give sufficient conditions under which a subgeometric completion is automatically lax-idempotent. In the latter subsections Section IV.3.3 and Section IV.3.4, we will demonstrate that the examples of subgeometric completions we have encountered so far satisfy this generalised definition.

Definition IV.33. Let A-**Doc** be a 2-full 2-subcategory of **Doc** (an object of A-**Doc** will be called an A-doctrine, and an arrow of A-**Doc** a morphism of A-doctrines) such that the image of the 2-functor

$$A\text{-}\mathbf{Doc}_{\text{flat}} \hookrightarrow \mathbf{Doc}_{\text{flat}} \xrightarrow{\mathfrak{Z}_{\text{Fr}}} \mathbf{GeomDoc} \subseteq \mathbf{Doc}_{\text{flat}}$$

is contained in A-**Doc**, as is the unit $\eta^{(P,J_{\text{triv}})} \colon P \to \mathfrak{Z}_{\text{Fr}}(P)$ for each A-doctrine $P \in A$-**Doc**. Here A-$\mathbf{Doc}_{\text{flat}}$ represents the 2-full 2-subcategory of A-**Doc** whose objects are A-doctrines and whose 1-cells are A-doctrine morphisms that are also flat. A 2-monad (T, ε, ν) on the 2-category A-**Doc**, thought of as a completion of A-doctrines, is said to be *subgeometric* if it satisfies the following conditions.

(i) For each A-doctrine $P \colon \mathcal{C}^{\text{op}} \to \mathbf{PreOrd}$ in A-**Doc**, there exists a choice of A-doctrine morphism
$$\xi_P \colon T\mathfrak{Z}_{\text{Fr}}(P) \longrightarrow \mathfrak{Z}_{\text{Fr}}(P)$$
such that $(\mathfrak{Z}_{\text{Fr}}(P), \xi_P)$ defines a T-algebra.

(ii) For each A-doctrine $P \colon \mathcal{C}^{\text{op}} \to \mathbf{PreOrd}$, there exists a Grothendieck topology J_P^T on the category $\mathcal{D} \rtimes TP$ such that:

   a) the unit $\varepsilon^P \colon P \to TP$ of the monad yields a morphism of doctrinal sites
   $$\varepsilon^P \colon (P, J_{\text{triv}}) \longrightarrow (TP, J_P^T),$$

   b) for each A-doctrine $P \colon \mathcal{C}^{\text{op}} \to \mathbf{PreOrd}$, the A-doctrine morphism ξ_P from above yields a morphism of doctrinal sites
   $$\xi_P \colon (T\mathfrak{Z}_{\text{Fr}}(P), J^T_{\mathfrak{Z}_{\text{Fr}}(P)}) \longrightarrow (\mathfrak{Z}_{\text{Fr}}(P), K_{\mathfrak{Z}_{\text{Fr}}(P)}),$$

   c) and the mapping that sends an A-doctrine $P \colon \mathcal{C}^{\text{op}} \to \mathbf{PreOrd}$ to the doctrinal site (TP, J_P^T) can be made functorial, i.e. each morphism of A-doctrines $\theta \colon P \to Q$ yields a morphism of doctrinal sites
   $$T\theta \colon (TP, J_P^T) \longrightarrow (TQ, J_Q^T).$$

   Thus there exists a 1-functor A-**Doc** $\to$ **DocSites** that acts on objects by $P \mapsto (TP, J_P^T)$ (we label this functor by J^T). In fact, since two morphisms $T\theta, T\theta' \colon TP \rightrightarrows TQ$ share the same 2-cells in both **Doc** and **DocSites**, J^T can be taken as a 2-functor.

Remark IV.34. (i) Condition (i) of Definition IV.33 expresses that the completion T completes an A-doctrine P to some fragment of the data of a geometric doctrine. Evidently, if $\mathfrak{Z}_{\mathrm{Fr}}(P)$ already possesses the structure which T is freely adding, then $\mathfrak{Z}_{\mathrm{Fr}}(P)$ is a T-algebra. Condition (ii) expresses that the added data can be 'seen' by a choice of Grothendieck topology.

(ii) In Definition IV.33, we also made the distinction between the category A-**Doc**, on which the monad of the subgeometric completion (T, ε, ν) acts, and the category A-$\mathbf{Doc}_{\mathrm{flat}}$. This pedantry is necessary to include as examples all the completions we would expect to be subgeometric. For example, the free top completion does not induce a monad on $\mathbf{Doc}_{\mathrm{flat}}$. For a preorder P, the inclusion $P \hookrightarrow P \oplus \top = P^{\top}$ of P into its free top completion, i.e. the unit of the completion, does not induce a morphism of doctrinal sites $(P, J_{\mathrm{triv}}) \to (P^{\top}, J_{\mathrm{triv}})$, but it does induce a morphism of doctrinal sites $(P, J_{\mathrm{triv}}) \to (P^{\top}, J^{\top}_{\mathrm{triv}})$ (see Lemma IV.12).

Theorem IV.35. *For each subgeometric completion (T, ε, ν) on a 2-subcategory A-**Doc** of **Doc**, the square*

$$\begin{tikzcd} A\text{-}\mathbf{Doc}_{\mathrm{flat}} \arrow[r, hook] \arrow[d, "J^T"'] & \mathbf{Doc}_{\mathrm{flat}} \arrow[d, "\mathfrak{Z}_{\mathrm{Fr}}"] \\ \mathbf{DocSites} \arrow[r, "\mathfrak{Z}"] & \mathbf{GeomDoc} \end{tikzcd}$$

*commutes up to 2-natural isomorphism. In particular, for each A-doctrine $P\colon C^{\mathrm{op}} \to$ **PreOrd**, there is an isomorphism $\mathfrak{Z}_{\mathrm{Fr}}(P) \cong \mathfrak{Z}(T(P), J^T_P)$.*

Proof. The component of the 2-natural isomorphism $\mathfrak{Z} \circ J^T \cong \mathfrak{Z}_{\mathrm{Fr}}$ at an A-doctrine P is given by $\mathfrak{Z}(\varepsilon^P)$ – that is the arrow

$$\begin{tikzcd} (P, J_{\mathrm{triv}}) \arrow[r, "\eta^{(P, J_{\mathrm{triv}})}"] \arrow[d, "\varepsilon^P"'] & (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \arrow[d, dashed, "\mathfrak{Z}(\varepsilon^P)"] \\ (TP, J^T_P) \arrow[r, "\eta^{(TP, J^T_P)}"] & (\mathfrak{Z}(TP, J^T_P), K_{\mathfrak{Z}(TP, J^T_P)}) \end{tikzcd}$$

as induced by the universal property of the geometric completion. By the 2-naturality of ε, it is trivial to see that the arrows $\mathfrak{Z}(\varepsilon^P)$ are the components of a 2-natural transformation.

It remains to show that $\mathfrak{Z}(\varepsilon^P)$ is an isomorphism for each A-doctrine $P \in A$-**Doc**. We exploit the universal property of the geometric completion to construct an inverse.

Consider the diagram

$$\begin{tikzcd}
 & (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \arrow[dr, bend left, "\mathfrak{Z}(\varepsilon^P)"] & \\
(P, J_{\mathrm{triv}}) \arrow[ur, bend left, "\eta^{(P,J_{\mathrm{triv}})}"] \arrow[r, "\varepsilon^P"] \arrow[d, "\varepsilon^P"'] \arrow[ddr, "\eta^{(P,J_{\mathrm{triv}})}"'] & (TP, J_P^T) \arrow[r, "\eta^{(TP,J_P^T)}"] \arrow[d, "T\eta^{(P,J_{\mathrm{triv}})}"] & (\mathfrak{Z}(TP, J_P^T), K_{\mathfrak{Z}(TP,J_P^T)}) \arrow[ddl, dashed, "\Xi_P"] \\
(TP, J_P^T) \arrow[d, "\eta^{(TP,J_P^T)}"'] & (T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \arrow[d, "\xi_P"] & \\
(\mathfrak{Z}(TP, J_P^T), K_{\mathfrak{Z}(TP,J_P^T)}) & (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}), \arrow[l, "\mathfrak{Z}(\varepsilon^P)"'] &
\end{tikzcd} \tag{IV.ix}$$

where the arrow $\Xi_P\colon (\mathfrak{Z}(TP, J_P^T), K_{\mathfrak{Z}(TP,J_P^T)}) \to (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})$ is induced by the universal property of the geometric completion.

We claim that the diagram (IV.ix) commutes – it suffices to only check that the triangle

$$\begin{tikzcd}
(P, J_{\mathrm{triv}}) \arrow[r, "\varepsilon^P"] \arrow[ddr, "\eta^{(P,J_{\mathrm{triv}})}"'] & (TP, J_P^T) \arrow[d, "T\eta^{(P,J_{\mathrm{triv}})}"] \\
 & (T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \arrow[d, "\xi_P"] \\
 & (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})
\end{tikzcd} \tag{IV.x}$$

commutes (the other sub-diagrams follow by definition). The triangle (IV.x) commutes since

$$\begin{aligned}
\xi_P \circ T\eta^{(P,J_{\mathrm{triv}})} \circ \varepsilon^P &= \xi_P \circ \varepsilon^{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ \eta^{(P,J_{\mathrm{triv}})} && \text{since } \varepsilon \text{ is natural,} \\
&= \eta^{(P,J_{\mathrm{triv}})} && \text{since } (\mathfrak{Z}_{\mathrm{Fr}}(P), \xi_P) \text{ is a } T\text{-algebra.}
\end{aligned}$$

Therefore, by the universal property of the geometric completion, we obtain the desired equations $\mathfrak{Z}(\varepsilon^P) \circ \Xi_P = \mathrm{id}_{\mathfrak{Z}(TP,J_P^T)}$ and $\Xi_P \circ \mathfrak{Z}(\varepsilon^P) = \mathrm{id}_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$. □

Remark IV.36. We saw in Section IV.1.1 that for each doctrine $P\colon C^{\mathrm{op}} \to \mathbf{PreOrd}$ and Grothendieck topology J on $C \rtimes P$, there is a Grothendieck topology $J^\top$ on the category $C \rtimes P^\top$, where $P^\top$ is the free (preserved) top completion, such that $\mathfrak{Z}(P, J) \cong \mathfrak{Z}(P^\top, J^\top)$. It is not hard to see that the notion of subgeometric completion and the result of Theorem IV.35 can be extended to encompass 2-subcategories A-$\mathbf{Doc} \subseteq \mathbf{DocSites}$ in addition to 2-subcategories A-$\mathbf{Doc} \subseteq \mathbf{Doc}$ as currently presented. We present the modified result below.

Let A-**Doc** be a 2-full 2-subcategory of **DocSites** endowed with a 2-monad (T, ε, ν). By $\mathbf{GeomDoc}_A$ denote the image of the composite

$$A\text{-}\mathbf{Doc} \hookrightarrow \mathbf{DocSites} \xrightarrow{\mathfrak{z}} \mathbf{GeomDoc}.$$

Suppose that, for each object $(P, J) \in A$-**Doc**, there exists a choice $J^A_{(P,J)}$ of a Grothendieck topology on the category $C \rtimes \mathfrak{z}(P, J)$ such that $(\mathfrak{z}(P, J), J^A_{(P,J)}) \in A$-**Doc** and $\mathfrak{z}(P, J)$ also satisfies the following properties.

(i) The choice of topology $J^T_{(P,J)}$ is 2-functorial, i.e. the action on objects that sends $\mathfrak{z}(P, J) \in \mathbf{GeomDoc}_A$ to the doctrinal site $(\mathfrak{z}(P, J), J^A_{(P,J)}) \in A$-**Doc** can be extended to a 2-functor
$$J^T : \mathbf{GeomDoc}_A \longrightarrow A\text{-}\mathbf{Doc}.$$

(ii) For each $(P, J) \in A$-**Doc**, there is a morphism
$$\xi_{(P,J)} : T(\mathfrak{z}(P, J), J^T_{(P,J)}) \longrightarrow (\mathfrak{z}(P, J), J^T_{(P,J)})$$
of A-**Doc** for which $((\mathfrak{z}(P, J), J^T_{(P,J)}), \xi_{(P,J)})$ is a T-algebra and, moreover, the underlying functor and natural transformation pair of $\xi_{(P,J)}$ define a morphism of doctrinal sites $\xi_{(P,J)} : T(\mathfrak{z}(P, J), J^T_{(P,J)}) \to (\mathfrak{z}(P, J), K_{\mathfrak{z}(P,J)})$.

Then the square

$$\begin{tikzcd}
A\text{-}\mathbf{Doc} \arrow[rr, hook] \arrow[d, "T"'] & & \mathbf{DocSites} \arrow[d, "\mathfrak{z}"] \\
A\text{-}\mathbf{Doc} \arrow[r, hook] & \mathbf{DocSites} \arrow[r, "\mathfrak{z}"] & \mathbf{GeomDoc}
\end{tikzcd}$$

commutes up to natural isomorphism.

When are subgeometric completions lax-idempotent? In Corollary IV.28, we observed that the free geometric completion $\mathfrak{z}_{\mathrm{Fr}}$ is lax-idempotent. We may wonder if this infers that any subgeometric completion is also lax-idempotent. The inference holds, under some further assumptions.

Proposition IV.37. *Let (T, ε, ν) be a subgeometric completion acting on A-**Doc**, such that*

(i) for each A-doctrine P, the natural transformation $\eta^{(P,J^T_P)} : TP \to \mathfrak{z}(TP, J^T_P)$ is pointwise injective,

(ii) and, for each A-doctrine P, the multiplication of the free geometric completion
$$\mu^P : \mathfrak{z}_{\mathrm{Fr}}\mathfrak{z}_{\mathrm{Fr}}(P) \longrightarrow \mathfrak{z}_{\mathrm{Fr}}(P)$$
yields a morphism $(\mathfrak{z}_{\mathrm{Fr}}\mathfrak{z}_{\mathrm{Fr}}(P), \xi_{\mathfrak{z}_{\mathrm{Fr}}(P)}) \to (\mathfrak{z}_{\mathrm{Fr}}(P), \xi_P)$ of T-algebras,

then (T, ε, ν) is lax-idempotent.

Proof. Recall that a 2-monad (τ, e, m) on $\mathcal{D}$ is lax-idempotent if, for each $d \in \mathcal{D}$, there is a 2-cell $\lambda_d \colon \tau e_d \Rightarrow e_{\tau d}$, natural in d, such that the two horizontal composites

$$\begin{tikzcd} d \arrow[r, "e_d"] & \tau d \arrow[r, bend left, "\tau e_d"] \arrow[r, bend right, "e_{\tau d}"'] \arrow[r, phantom, "\Downarrow \lambda_d"] & \tau\tau d, \end{tikzcd} \quad \begin{tikzcd} \tau d \arrow[r, bend left, "\tau e_d"] \arrow[r, bend right, "e_{\tau d}"'] \arrow[r, phantom, "\Downarrow \lambda_d"] & \tau\tau d \arrow[r, "m_d"] & \tau d, \end{tikzcd}$$

i.e. $\lambda_d * e_d$ and $m_d * \lambda_d$, are the identity 2-cells.

Our strategy for the proof is to lift the 2-cell $\lambda_P \colon \mathfrak{Z}_{\mathrm{Fr}}(\eta^{(P,J_{\mathrm{triv}})}) \Rightarrow \eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P),J_{\mathrm{triv}})}$, corresponding to an A-doctrine P, to a 2-cell $\lambda_P \colon T\varepsilon^P \Rightarrow \varepsilon^{TP}$. We will then use that the free geometric completion $\mathfrak{Z}_{\mathrm{Fr}}$ is lax idempotent, i.e. that $\lambda_P * \eta^{(P,J_{\mathrm{triv}})}$ and $\mu^P * \lambda_P$ are both the identity 2-cells, to deduce the corresponding equations for the 2-monad T.

Since for each A-doctrine P, the natural transformation $\eta^{(P,J_P^T)} \colon TP \to \mathfrak{Z}(TP, J_P^T)$ is pointwise injective, by Remark IV.20 the functor

$$\mathbf{DocSites}((TP, J_P^T), (TTP, J_{TP}^T)) \longrightarrow \mathbf{GeomDoc}\Big(\mathfrak{Z}(TP, J_P^T), \mathfrak{Z}(\mathfrak{Z}(TP, J_P^T), J^T_{\mathfrak{Z}(TP,J_P^T)})\Big)$$

induced by $\eta^{(TP,J_P^T)}$ is full and faithful. Hence, so too is the functor

$$\mathbf{DocSites}((TP, J_P^T), (TTP, J_{TP}^T)) \longrightarrow \mathbf{GeomDoc}(\mathfrak{Z}_{\mathrm{Fr}}(P), \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)) \tag{IV.xi}$$

induced by the composite $\Xi_P \circ \eta^{(TP,J_P^T)}$, where Ξ_P is the inverse to $\mathfrak{Z}(\varepsilon^P)$ as constructed in Theorem IV.35. We will write Θ_P for the composite $\Xi_P \circ \eta^{(TP,J_P^T)} \colon TP \to \mathfrak{Z}_{\mathrm{Fr}}(P)$.

Therefore, since $\mathfrak{Z}_{\mathrm{Fr}}$ is lax-idempotent, the corresponding 2-cell $\lambda_P \colon \mathfrak{Z}_{\mathrm{Fr}}(\eta^{(P,J_{\mathrm{triv}})}) \Rightarrow \eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P),J_{\mathrm{triv}})}$ lifts, as (IV.xi) is full, to a 2-cell $\lambda_P \colon T\varepsilon^P \Rightarrow \varepsilon^{TP}$ (this 2-cell is of course labelled by id_C, where P is fibred over the category C – but that should not be confused with it being the identity 2-cell).

Note that, by definition, for each A-doctrine P the diagram

$$\begin{tikzcd} & TP \arrow[d, "\eta^{(TP,J_P^T)}"'] \arrow[dd, bend left=60, "\Theta_P"] \\ P \arrow[ur, bend left, "\varepsilon^P"] \arrow[dr, bend right, "\eta^{(P,J_{\mathrm{triv}})}"'] & \mathfrak{Z}(TP, J_P^T) \arrow[d, shift left, "\Xi_P"] \\ & \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[u, shift left, "\mathfrak{Z}_{\mathrm{Fr}}(\varepsilon^P)"] \end{tikzcd}$$

commutes. Therefore, the diagram

$$\begin{tikzcd}
P \arrow[r, "\varepsilon^P"] \arrow[d, equal] & TP \arrow[rr, bend left, "T\varepsilon^P"] \arrow[rr, bend right, "\varepsilon^{TP}"'] \arrow[rr, phantom, "\Downarrow \lambda_P"] \arrow[d, "\Theta_P"'] & & TTP \arrow[d, "\Theta_{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ T\Theta_P"] \\
P \arrow[r, "\eta^{(P, J_{\mathrm{triv}})}"'] & \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[rr, bend left, "\mathfrak{Z}_{\mathrm{Fr}}\left(\eta^{(P, J_{\mathrm{triv}})}\right)"] \arrow[rr, bend right, "\eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P), J_{\mathrm{triv}})}"'] \arrow[rr, phantom, "\Downarrow \lambda_P"] & & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P),
\end{tikzcd}$$

also commutes. Since

$$\mathbf{DocSites}((P, J_{\mathrm{triv}}), (TTP, J^T_{TP})) \longrightarrow \mathbf{DocSites}((P, J_{\mathrm{triv}}), (\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)}))$$

is faithful, again by Remark IV.20, we conclude that $\lambda_P * \varepsilon^P$ is indeed the identity 2-cell.

We exploit a symmetric argument to conclude that $\nu^P * \lambda_P$ is also the identity 2-cell. We claim that the diagram

$$\begin{tikzcd}
TP \arrow[rr, bend left, "T\varepsilon^P"] \arrow[rr, bend right, "\varepsilon^{TP}"'] \arrow[rr, phantom, "\Downarrow \lambda_P"] \arrow[d, "\Theta_P"'] & & TTP \arrow[r, "\nu^P"] \arrow[d, "\Theta_{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ T\Theta_P"] & TP \arrow[d, "\Theta_P"] \\
\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[rr, bend left, "\mathfrak{Z}_{\mathrm{Fr}}\left(\eta^{(P, J_{\mathrm{triv}})}\right)"] \arrow[rr, bend right, "\eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P), J_{\mathrm{triv}})}"'] \arrow[rr, phantom, "\Downarrow \lambda_P"] & & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\mu^P"] & \mathfrak{Z}_{\mathrm{Fr}}(P)
\end{tikzcd} \tag{IV.xii}$$

also commutes. Using that

$$\mathbf{DocSites}((TP, J^T_P), (TP, J^T_P)) \longrightarrow \mathbf{GeomDoc}(\mathfrak{Z}_{\mathrm{Fr}}(P), \mathfrak{Z}_{\mathrm{Fr}}(P))$$

is faithful, again by Remark IV.20, we conclude that $\nu^P * \lambda_P$ is the identity 2-cell as desired.

However, demonstrating the commutativity of (IV.xii), that is the commutativity of the required square

$$\begin{tikzcd}
TTP \arrow[r, "\nu^P"] \arrow[d, "\Theta_{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ T\Theta_P"'] & TP \arrow[d, "\Theta_P"] \\
\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\mu^P"] & \mathfrak{Z}_{\mathrm{Fr}}(P),
\end{tikzcd} \tag{IV.xiii}$$

is more involved. First, recall from Theorem IV.35 that, for each A-doctrine P, the square

$$
\begin{tikzcd}
TP \arrow[r, "\eta^{(TP,J^T_P)}"] \arrow[d, "T\eta^{(P,J_{\mathrm{triv}})}"] & \mathfrak{Z}(TP, J^T_P) \arrow[d, "\Xi_P"] \\
T\mathfrak{Z}_{\mathrm{Fr}} \arrow[r, "\xi_P"] & \mathfrak{Z}_{\mathrm{Fr}}(P)
\end{tikzcd}
\tag{IV.xiv}
$$

commutes. We can show that the square (IV.xiii) commutes by decomposing it as

$$
\begin{tikzcd}
TTP \arrow[rrrrr, "\nu^P"] \arrow[dd, "T\eta^{(TP,J^T_P)}"'] \arrow[ddrr, "TT\eta^{(P,J_{\mathrm{triv}})}"] & & & & & TP \arrow[ddll, "T\eta^{(P,J_{\mathrm{triv}})}"'] \arrow[dd, "\eta^{(TP,J^T_P)}"] \\
& & ④ & & & \\
T\mathfrak{Z}(TP, J^T_P) \arrow[dd, "T\Xi_p"'] & ③ & TT\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\nu^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \arrow[ddll, "T\xi_P"] & T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[ddrr, "\xi_P"] & ① & \mathfrak{Z}(TP, J^T_P) \arrow[dd, "\Xi_P"] \\
& & ⑤ & & & \\
T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[rrrrr, "\xi_P"] \arrow[dd, "\eta^{(T\mathfrak{Z}_{\mathrm{Fr}}(P),J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)})}"'] \arrow[ddrr, "T\eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P),J_{\mathrm{triv}})}"] & & & & & \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[dddd, equals] \\
& & & & & \\
\mathfrak{Z}(T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \arrow[dd, "\Xi_{\mathfrak{Z}_{\mathrm{Fr}}(P)}"'] & ② & T\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "T\mu^P"] \arrow[ddll, "\xi_{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] & T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[ddrr, "\xi_P"] & & \\
& & ⑥ & & & \\
\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[rrrrr, "\mu^P"] & & & & & \mathfrak{Z}_{\mathrm{Fr}}(P).
\end{tikzcd}
$$

The squares ① and ② commute by (IV.xiv), and the square ③ is just T applied to (IV.xiv). The square ④ commutes by the naturality of $\nu \colon TT \to T$. The square ⑤ commutes since $(\mathfrak{Z}_{\mathrm{Fr}}(P), \xi_P)$ is a T-algebra. The square ⑥ commutes by the assumption that μ^P yields a morphism $(\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), \xi_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \to (\mathfrak{Z}_{\mathrm{Fr}}(P), \xi_P)$ of T-algebras. Finally, the remaining equation to check, that

$$\xi_P \circ T\mu^P \circ T\eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P),J_{\mathrm{triv}})} = \xi_P,$$

follows from $\mu^P \circ \eta^{(\mathfrak{Z}_{\mathrm{Fr}}(P),J_{\mathrm{triv}})} = \mathrm{id}_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$, the unit law for $(\mathfrak{Z}_{\mathrm{Fr}}, \eta, \mu)$. □

IV.3.3 Subgeometric completions via subdoctrines

Let $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ be a primary doctrine. The statement of Proposition IV.31 expresses that the fibre at c of the geometric completion $P^{\exists}(c)$ can be recovered as the subset of $\mathfrak{Z}_{\mathrm{Fr}}(P)(c)$, specifically as the subset of elements of the form $\exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)$, where $d \xrightarrow{f} c$ is an arrow of $\mathcal{C}$ and $x \in P(d)$. Furthermore, the elements $\exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x) \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ can be characterised as the *supercompact* objects of the site $(\mathcal{C} \rtimes \mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})$.

Lemma IV.38. *An element $S \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ is of the form $\exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)$ if and only if (c,S) is* supercompact, *i.e. every $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-cover of (c,S) contains a singleton subcover.*

Proof. Firstly, if S is supercompact then, since (c,S) admits the $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-cover

$$\left\{ (d, \eta_d^{(P,J_{\mathrm{triv}})}(x)) \xrightarrow{f} (c,S) \,\middle|\, (f,x) \in S \right\},$$

S must be equal to $\exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)$ for some $(f,x) \in S$.

The object $(c, \exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x))$ is supercompact since if

$$\left\{ (e_i, T_i) \xrightarrow{g_i} (c, \exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)) \,\middle|\, i \in I \right\}$$

is a $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-cover, then $\bigcup_{i\in I} \exists_{g_i} T_i = \exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)$, and so $(f,x) \in \exists_{g_{i'}} T_{i'}$ for some $i' \in I$. Therefore, $\exists_{g_{i'}} T_{i'} = \exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)$ and so the singleton arrow

$$\left\{ (e_{i'}, T_{i'}) \xrightarrow{g_{i'}} (c, \exists_f \eta_d^{(P,J_{\mathrm{triv}})}(x)) \right\}$$

is a $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-cover. □

In this subsection we study completions of doctrines obtained in an analogous fashion by taking certain subdoctrines of the free geometric completion. We will formulate a general theory for such completions obtained via subdoctrines, and demonstrate that they are subgeometric in the sense of Definition IV.33, thus providing a broad class of examples of subgeometric completions. Moreover, we show that the induced 2-monads are all lax-idempotent.

Among the examples of subgeometric completions we are able to obtain in this way is the existential completion $T^{\exists}\colon \mathbf{PrimDoc} \to \mathbf{PrimDoc}$ established in [119]. We will also obtain a lax-idempotent *free coherent completion* for primary doctrines. Finally, we will relate the existential and coherent completions thus obtained to the regular and coherent completions of cartesian categories.

Compatible subcompletions. We first develop our general theory for completions of doctrines obtained via subdoctrines of the free geometric completion. We call these *compatible subcompletions* in analogy with the terminology 'compatible properties' used in the topos-theoretic study of Stone-type dualities given in [19, §3]. Given a doctrine $Q\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$, by a *subdoctrine* of Q we mean a doctrine $Q'\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$, also indexed over $\mathcal{C}$, and a natural transformation $Q' \hookrightarrow Q$ for which every component is a subset inclusion $Q'(c) \subseteq Q(c)$.

For this subsection, in every doctrinal site (P,J) we encounter, the topology J is taken to be the trivial topology J_{triv}. Therefore, we abbreviate our notation and write η^P for $\eta^{(P,J_{\mathrm{triv}})}$, μ^P for $\mu^{(P,J_{\mathrm{triv}})}$, etc.

Definition IV.39. Let A-**Doc** be a 2-full 2-subcategory of $\mathbf{Doc}_{\mathrm{flat}}$ that contains the image of the functor

$$A\text{-}\mathbf{Doc} \hookrightarrow \mathbf{Doc}_{\mathrm{flat}} \xrightarrow{\mathfrak{Z}_{\mathrm{Fr}}} \mathbf{GeomDoc} \subseteq \mathbf{Doc}_{\mathrm{flat}},$$

as well as the unit $\eta^P\colon P \to \mathfrak{Z}_{\mathrm{Fr}}(P)$ for each A-doctrine $P \in A$-**Doc**. A choice of a subdoctrine $H^P\colon TP \hookrightarrow \mathfrak{Z}_{\mathrm{Fr}}(P)$, for each A-doctrine P, is said to be *A-compatible* if the following conditions are satisfied.

(i) For each A-doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$, TP is a subdoctrine of $\mathfrak{Z}_{\mathrm{Fr}}(P)$ that contains the image of the unit η^P, i.e. there is a factorisation

$$\begin{tikzcd} P \arrow[r, "\varepsilon^P"] \arrow[rr, bend left, "\eta^P"] & TP \arrow[r, hook, "H^P"] & \mathfrak{Z}_{\mathrm{Fr}}(P). \end{tikzcd}$$

We also require that the factoring morphism $\varepsilon^P\colon P \to TP$ is a morphism of A-doctrines.

(ii) The choice of subdoctrine is natural in the sense that, for each morphism of A-doctrines $(F,a)\colon P \to Q$, the induced morphism of geometric doctrines $\mathfrak{Z}_{\mathrm{Fr}}(F,a)\colon \mathfrak{Z}_{\mathrm{Fr}}(P) \to \mathfrak{Z}_{\mathrm{Fr}}(Q)$ restricts to $TP \to TQ$, as in the diagram

$$\begin{tikzcd} P \arrow[r, "\varepsilon^P"] \arrow[d, "(F,a)"'] & TP \arrow[r, hook, "H^P"] \arrow[d, dashed] & \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[d, "\mathfrak{Z}_{\mathrm{Fr}}(F,a)"] \\ Q \arrow[r, "\varepsilon^Q"] & TQ \arrow[r, hook, "H^Q"] & \mathfrak{Z}_{\mathrm{Fr}}(Q). \end{tikzcd} \tag{IV.xv}$$

Moreover, we also require that the restriction $TP \to TQ$ is a morphism of A-doctrines, and so we obtain an (1-)endofunctor $T\colon A\text{-}\mathbf{Doc} \to A\text{-}\mathbf{Doc}$.

(iii) For each $P \in A\text{-}\mathbf{Doc}$, the subdoctrine $H^P\colon TP \hookrightarrow \mathfrak{Z}_{\mathrm{Fr}}(P)$ is 'compatible' with the multiplication of the free geometric completion $\mu^P\colon \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \to \mathfrak{Z}_{\mathrm{Fr}}(P)$ in the sense that the composite

$$TTP \xrightarrow{\ TH^P\ } T\mathfrak{Z}_{\mathrm{Fr}}(P) \xrightarrow{\ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}\ } \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \xrightarrow{\ \mu^P\ } \mathfrak{Z}_{\mathrm{Fr}}(P) \tag{IV.xvi}$$

factors through the subdoctrine $H^P\colon TP \hookrightarrow \mathfrak{Z}_{\mathrm{Fr}}(P)$, and this factorisation

$$\nu^P\colon TTP \longrightarrow TP$$

is a morphism of A-doctrines.

Examples IV.40. There are two basic examples to keep in mind for motivating our development. In both cases, the 2-category A-**Doc** is taken to be the 2-category of primary doctrines **PrimDoc**.

(i) The first example has been encountered already. For each primary doctrine P, taking $T^{\exists}P \hookrightarrow \mathfrak{Z}_{\mathrm{Fr}}(P)$ as the subdoctrine on supercompact elements is **PrimDoc**-compatible. While not every morphism of geometric doctrines

$$(G,b)\colon \mathfrak{Z}_{\mathrm{Fr}}(P) \longrightarrow \mathfrak{Z}_{\mathrm{Fr}}(Q)$$

sends a supercompact element $S \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ to a supercompact element $b_c(S)$ of $\mathfrak{Z}_{\mathrm{Fr}}(Q)(G(c))$, this is however true for morphisms of the form $\mathfrak{Z}_{\mathrm{Fr}}(F,a)$, where $(F,a)\colon P \to Q$ is a morphism of primary doctrines.

An element of $TTP(c)$ is of the form

$$\exists_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)(g)}\eta_e^{\mathfrak{Z}_{\mathrm{Fr}}(P)}\left(\exists_{\mathfrak{Z}_{\mathrm{Fr}}(P)(f)}\eta_d^P(x)\right),$$

for a composable pair of arrows $d \xrightarrow{f} e, e \xrightarrow{g} c \in C$ and an element $x \in P(d)$. One can calculate that

$$\mu_e^P\left(\exists_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)(g)}\eta_e^{\mathfrak{Z}_{\mathrm{Fr}}(P)}\left(\exists_{\mathfrak{Z}_{\mathrm{Fr}}(P)(f)}\eta_d^P(x)\right)\right) = \exists_{\mathfrak{Z}_{\mathrm{Fr}}(P)(g\circ f)}\eta_d^P(x).$$

Thus, μ^P restricts to a morphism $\nu^P\colon TTP \to TP$. The other required conditions on $T^{\exists}$ are easily checked.

(ii) Now consider taking $T^{\mathrm{Coh}}P$ to the be the subdoctrine of $\mathfrak{Z}_{\mathrm{Fr}}(P)$ on *compact elements*, i.e. $T^{\mathrm{Coh}}P(c)$ are those elements $S \in \mathfrak{Z}_{\mathrm{Fr}}(P)(c)$ such that every $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-cover of (c, S) has a finite subcover. Checking that this choice of subdoctrine of $\mathfrak{Z}_{\mathrm{Fr}}(P)$ satisfies the required conditions is analogous to the case for $T^{\exists}$.

The (1-)endofunctor $T\colon A\text{-}\mathbf{Doc} \to A\text{-}\mathbf{Doc}$ is evidently 2-functorial. Every 2-cell $\alpha\colon (F, a) \Rightarrow (F', a')$ (i.e. a suitable natural transformation $\alpha\colon F \Rightarrow F'$) between A-doctrine morphisms $(F, a), (F', a')\colon P \rightrightarrows Q$ yields a 2-cell

$$\begin{tikzcd} \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, bend left, "\mathfrak{Z}_{\mathrm{Fr}}(F,a)"] \arrow[r, bend right, "\mathfrak{Z}_{\mathrm{Fr}}(F',a')"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathfrak{Z}_{\mathrm{Fr}}(Q) \end{tikzcd}$$

since $\mathfrak{Z}_{\mathrm{Fr}}$ is 2-functorial. Therefore, α also defines a 2-cell between the restrictions to the subdoctrines

$$\begin{tikzcd} TP \arrow[r, bend left, "T(F,a)"] \arrow[r, bend right, "T(F',a')"'] \arrow[r, phantom, "\Downarrow\alpha"] & TQ. \end{tikzcd}$$

We note also that, since the triple (T, ε, ν) is a restriction of the 2-monad $(\mathfrak{Z}_{\mathrm{Fr}}, \eta, \mu)$, the monad equations for (T, ε, ν) follow automatically.

Lemma IV.41. *The triple (T, ε, ν) is a 2-monad on A-$\mathbf{Doc}$.*

Definition IV.42. We call this 2-monad the *compatible subcompletion*.

Proposition IV.43. *Every compatible subcompletion $T\colon A$-$\mathbf{Doc} \to A$-$\mathbf{Doc}$ is subgeometric.*

Proof. For each A-doctrine P, the morphism

$$T\mathfrak{Z}_{\mathrm{Fr}}(P) \xrightarrow{H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}} \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \xrightarrow{\mu^P} \mathfrak{Z}_{\mathrm{Fr}}(P)$$

is a natural way of endowing $\mathfrak{Z}_{\mathrm{Fr}}(P)$ with the structure of a T-algebra. The unit condition, i.e. the commutativity of the triangle

$$\begin{tikzcd} \mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\varepsilon^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \arrow[ddr, equal] & T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[d, "H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \\ & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[d, "\mu^P"] \\ & \mathfrak{Z}_{\mathrm{Fr}}(P), \end{tikzcd}$$

is satisfied since
$$\mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ \varepsilon^{\mathfrak{Z}_{\mathrm{Fr}}(P)} = \mu^P \circ \eta^{\mathfrak{Z}_{\mathrm{Fr}}(P)} = \mathrm{id}_{\mathfrak{Z}_{\mathrm{Fr}}(P)}.$$
The action property, i.e. that
$$\mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ T(\mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}) = \mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ \nu^{\mathfrak{Z}_{\mathrm{Fr}}(P)},$$
follows from the commutativity of the diagram

$$\begin{tikzcd}
TT\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "TH^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \arrow[dd, "\nu^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"'] & T\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "T\mu^P"] \arrow[d, "H^{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \arrow[dr, phantom, "②"] & T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[d, "H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \\
③ & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\mathfrak{Z}_{\mathrm{Fr}}\mu^P"] \arrow[d, "\mu^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] \arrow[dr, phantom, "①"] & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[d, "\mu^P"] \\
T\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "H^{\mathfrak{Z}_{\mathrm{Fr}}(P)}"] & \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P) \arrow[r, "\mu^P"] & \mathfrak{Z}_{\mathrm{Fr}}(P).
\end{tikzcd} \tag{IV.xvii}$$

The commutativity of the square ① is assured since $(\mathfrak{Z}_{\mathrm{Fr}}, \eta, \mu)$ is a 2-monad, while the squares ② and ③ commute by definition (see the equations (IV.xv) and (IV.xvi)).

We now seek to find a Grothendieck topology J_P^T on $\mathcal{C} \rtimes TP$ satisfying the required conditions of Definition IV.33. We take the obvious choice: $\mathcal{C} \rtimes TP$ is a subcategory of $\mathcal{C} \rtimes \mathfrak{Z}_{\mathrm{Fr}}(P)$ since TP is a subdoctrine of $\mathfrak{Z}_{\mathrm{Fr}}(P)$, and so we define J_P^T as the restriction of $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$ to $\mathcal{C} \rtimes P$. We check that the three conditions of Definition IV.33(ii) are satisfied.

(a) Recall that that the unit of the free geometric completion yields a dense morphism of sites $\mathrm{id}_{\mathcal{C}} \rtimes \eta^P : (\mathcal{C} \rtimes P, J_{\mathrm{triv}}) \to (\mathcal{C} \rtimes \mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})$. The functor $\mathrm{id}_{\mathcal{C}} \rtimes \eta^P$ factorises as
$$(\mathcal{C} \rtimes P, J_{\mathrm{triv}}) \xrightarrow{\mathrm{id}_{\mathcal{C}} \rtimes \varepsilon^P} (\mathcal{C} \rtimes TP, J_P^T) \xrightarrow{\mathrm{id}_{\mathcal{C}} \rtimes H^P} (\mathcal{C} \rtimes \mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}).$$
The right factor, $\mathrm{id}_{\mathcal{C}} \rtimes H^P$, is the inclusion of a dense subcategory, and hence also a dense morphism of sites. Therefore, by [107, Corollary 11.6], $\mathrm{id}_{\mathcal{C}} \rtimes \varepsilon^P$ is a morphism of sites, and so $\varepsilon^P : (P, J_{\mathrm{triv}}) \to (TP, J_P^T)$ is a morphism of doctrinal sites as desired.

(b) Firstly, the functor μ^P defines a morphism of doctrinal sites
$$\mu^P : (\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)}) \longrightarrow (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})$$
Secondly, $\mathrm{id}_{\mathcal{C}} \rtimes H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} : (\mathcal{C} \rtimes T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \to (\mathcal{C} \rtimes \mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)})$ is the inclusion of a dense subcategory, and therefore
$$H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} : (T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \longrightarrow (\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)})$$
is also a morphism of doctrinal sites. Hence, the composite
$$\mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)} : (T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \longrightarrow (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)})$$
defines a morphism of doctrinal sites as required.

(c) Finally, we wish to show that $T\theta\colon (TP, J^T_P) \to (TQ, J^T_Q)$ is a morphism of doctrinal sites for each morphism of A-doctrines $\theta\colon P \to Q$. By assumption, $T\theta$ is already flat by virtue of being a morphism of A-doctrines. That $T\theta$ sends J^T_P-covers to J^T_Q covers follows since $\mathfrak{Z}_{\mathrm{Fr}}(\theta)\colon \mathfrak{Z}_{\mathrm{Fr}}(P) \to \mathfrak{Z}_{\mathrm{Fr}}(Q)$ sends $K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}$-covers to $K_{\mathfrak{Z}_{\mathrm{Fr}}(Q)}$-covers.

Hence, all the conditions of Definition IV.33 are satisfied. □

An application of Proposition IV.37 now shows that every compatible subcompletion is lax-idempotent. The two conditions of Proposition IV.37 are clearly satisfied.

(i) Each component $\eta_c^{(TP,J^T_P)}\colon TP(c) \to \mathfrak{Z}(TP, J^T_P)(c)$ is injective – indeed it is isomorphic to the inclusion $H^P_c\colon TP(c) \hookrightarrow \mathfrak{Z}_{\mathrm{Fr}}(P)(c) \cong \mathfrak{Z}(TP, J^T_P)(c)$.

(ii) Secondly, for each A-doctrine P,

$$\mu^P\colon (\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P), \mu^{\mathfrak{Z}_{\mathrm{Fr}}(P)} \circ H^{\mathfrak{Z}_{\mathrm{Fr}}\mathfrak{Z}_{\mathrm{Fr}}(P)}) \longrightarrow (\mathfrak{Z}_{\mathrm{Fr}}(P), \mu^P \circ H^{\mathfrak{Z}_{\mathrm{Fr}}(P)})$$

is a morphism of T-algebras by the commutativity of the right hand side of the diagram (IV.xvii).

Corollary IV.44. *Every compatible subcompletion (T, ε, ν) is lax-idempotent.*

The regular and coherent completions. Let us revisit the examples of compatible subcompletions given in Examples IV.40. As remarked in Lemma IV.38, we have recovered the existential completion established in [119], the lax-idempotent 2-monad $T^\exists\colon \mathbf{PrimDoc} \to \mathbf{PrimDoc}$, as a compatible subcompletion.

The 2-category of algebras for the 2-monad $T^\exists$ is precisely the the 2-category **ExDoc** of existential doctrines (see [119, Corollary 5.5]). In a similar fashion, we recognise the 2-category of algebras for the lax-idempotent 2-monad $T^{\mathrm{Coh}}\colon \mathbf{PrimDoc} \to \mathbf{PrimDoc}$ as the 2-category **CohDoc** of coherent doctrines. Using the inherent 2-adjunction

$$\begin{tikzcd} \tau\text{-}\mathbf{Alg} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\tau"'] \end{tikzcd}$$

for a 2-monad (τ, e, m) on a 2-category $\mathcal{C}$, we recover the following completions of doctrines.

Corollary IV.45 (§5 [119])**.** *(i) The 2-embedding* **ExDoc** $\hookrightarrow$ **PrimDoc** *possesses a lax-idempotent left 2-adjoint.*

(ii) The 2-embedding **CohDoc** $\hookrightarrow$ **PrimDoc** *possesses a lax-idempotent left 2-adjoint.*

The regular and coherent completions of cartesian categories. Following the example of [83], we turn to using these completions of doctrines to describe completions of categories. We have seen in Corollary IV.30 that the free geometric completion yields the completion of a cartesian category to a geometric category. We deduce that, in a similar manner, the subgeometric completions we have constructed in this subsection yield other completions of cartesian categories.

As already noted in [119, §6], the existential completion of a primary doctrine can be used to recover the *regular completion* of a cartesian category. For a cartesian category $\mathcal{C}$, Carboni describes in [27] the regular completion $\mathbf{Reg}(\mathcal{C})$ as follows:

(i) the objects of $\mathbf{Reg}(C)$ are arrows $d \xrightarrow{f} c$ of C;

(ii) an arrow $[g]\colon f_1 \to f_2$ of $\mathbf{Reg}(C)$ is an equivalence class of arrows $d_1 \xrightarrow{g} d_2$ such that

$$\begin{tikzcd} e \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & d_1 \arrow[r, "f_2 \circ g"] & c_2 \end{tikzcd}$$

commutes, where (h, k) are the kernel pair of f_1, i.e.

$$\begin{tikzcd} e \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & d_1 \arrow[d, "f_1"] \\ d_1 \arrow[r, "f_1"] & c_1 \end{tikzcd}$$

is a pullback. Two such arrows

$$\begin{tikzcd} d_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & d_2 \end{tikzcd}$$

are equivalent, i.e. $[g] = [g']$, if

$$\begin{tikzcd} d_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & d_2 \arrow[r, "f_2"] & c_2 \end{tikzcd}$$

commutes.

In [27, §5] it is shown that this defines the action on objects of a pseudo-adjoint to the 2-embedding $\mathbf{Reg} \hookrightarrow \mathbf{Cart}$ of cartesian categories into regular categories. In an analogous manner to Corollary IV.30, we deduce that $\mathbf{Syn}(T^{\exists}\mathrm{Sub}_C)$ satisfies the same universal property as $\mathbf{Reg}(C)$, and hence $\mathbf{Syn}(T^{\exists}\mathrm{Sub}_C) \simeq \mathbf{Reg}(C)$. Similarly, by considering the category $\mathbf{Syn}(T^{\mathrm{Coh}}\mathrm{Sub}_C)$, for a cartesian category C, we obtain the universal *coherent completion* of C.

Corollary IV.46. *The 2-embedding* **Coh** $\hookrightarrow$ **Cart** *has a left pseudo-adjoint – the* coherent completion *of a cartesian category.*

IV.3.4 Pointwise subgeometric completions

In this final subsection, we revisit the free top completion as a subgeometric completion in light of Definition IV.33. Since the syntax of geometric logic is often represented by the symbols $\{\top, \exists, =, \bigvee, \wedge\}$, we 'complete the set', so to speak, by also briefly sketching that the *free join* and *free binary meet* completions also constitute subgeometric completions. The completion with respect to the either of the symbols $\exists$ and $=$ is the previously discussed existential completion – for $\exists$, we freely add left adjoints to product projections, while for $=$ we freely add left adjoints to diagonals (both are subgeometric, see Remark IV.32). In what follows, the conditions of Definition IV.33 are easily, but tediously, checked – and so we omit many of the details.

Since the completions we consider in this subsection are of a 'pointwise' nature, we first state some easily deduced facts concerning such completions. Suppose that

A-**PreOrd** is a 2-full 2-subcategory of **PreOrd** whose inclusion $A\text{-}\mathbf{PreOrd} \hookrightarrow \mathbf{PreOrd}$ has a (strict) left 2-adjoint $T^A \colon \mathbf{PreOrd} \to A\text{-}\mathbf{PreOrd}$. Equivalently, for each preorder P, the completion $T^A P$ has the universal property that for any monotone map $a \colon P \to Q$ whose codomain lies in A-**PreOrd**, there is a unique morphism $a^A \colon T^A P \to Q$ of A-**PreOrd** for which the triangle

$$\begin{tikzcd} P \arrow[r, "\varepsilon"] \arrow[dr, "a"'] & T^A P \arrow[d, dashed, "a^A"] \\ & Q \end{tikzcd}$$

commutes (where ε is the unit of the 2-adjunction). It is clearly deduced that the functor T^A extends to a (strict) 2-adjunction

$$\begin{tikzcd} {[\mathcal{C}^{\mathrm{op}}, \mathbf{PreOrd}]} \arrow[r, shift left, "T'^A"] \arrow[r, phantom, "\perp"] & {[\mathcal{C}^{\mathrm{op}}, A\text{-}\mathbf{PreOrd}]}, \arrow[l, shift left, hook'] \end{tikzcd}$$

and hence also a (strict) 2-adjunction

$$\begin{tikzcd} \mathbf{Doc} \arrow[r, shift left, "T''^A"] \arrow[r, phantom, "\perp"] & A\text{-}\mathbf{Doc}, \arrow[l, shift left, hook'] \end{tikzcd}$$

where A-**Doc** is the category of A-**PreOrd**-valued doctrines.

Free top completion. Let $T^\top \colon \mathbf{Doc} \to \mathbf{Doc}$ denote the free (preserved) top completion monad constructed in Section IV.1.1. Having preserved top elements, every geometric doctrine $\mathbb{L} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$ can naturally be turned into an algebra for the monad $T^\top$. We have also already encountered the topology $J^\top_{\mathrm{triv}}$ on the category $\mathcal{C} \rtimes P^\top$, where P is a doctrine $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$. That this choice of Grothendieck topology satisfies the condition of Definition IV.33 is easily shown: for example, that the unit $(P, J_{\mathrm{triv}}) \hookrightarrow (P^\top, J^\top_{\mathrm{triv}})$ is a morphism of sites follows from Lemma IV.12. Thus, we can apply Theorem IV.35 to deduce that $T^\top$ is subgeometric, yielding a 'top-down' proof of Proposition IV.13.

Free join completion. As previously mentioned, the 2-functor

$$\mathfrak{Z}_{\mathrm{Ex}} \colon \mathbf{ExDoc} \longrightarrow \mathbf{GeomDoc}$$

sends an existential doctrine $P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{MSLat}$ to its 'pointwise' join completion $2^{(-)^{\mathrm{op}}} \circ P \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Frm}_{\mathrm{open}}$, i.e. $\mathfrak{Z}_{\mathrm{Ex}}(P)(c)$ is the poset of down-sets of $P(c)$ ordered by inclusion.

We could also conceive of taking the 'pointwise' join completion $2^{(-)^{\mathrm{op}}} \circ P$ of any doctrine $P \in \mathbf{Doc}$. Hence an element $J \in 2^{(-)^{\mathrm{op}}} \circ P(c)$ is a down-set of $P(c)$. By the above discussion, this yields a left adjoint $T^\vee$ to the inclusion of **SupSLat**-valued doctrines into **Doc**, where **SupSLat** is the 2-category of sup-semilattices (i.e. posets with all joins), their homomorphisms, and natural transformations between these. By the universal property of $T^\vee$, for each geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$ there exists a natural transformation $\mathrm{id}^\vee_{\mathbb{L}} \colon T^\vee \mathbb{L} \Rightarrow \mathbb{L}$ for which $(\mathbb{L}, \mathrm{id}^\vee_{\mathbb{L}})$ is a $T^\vee$-algebra.

For each doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$, the choice of the topology $J_P^{\vee}$, where $J_P^{\vee}$ is the Grothendieck topology on $\mathcal{C} \rtimes T^{\vee}P$ generated by covering families of the form

$$\left\{ (c, J_i) \xrightarrow{\mathrm{id}_c} \left(c, \bigcup_{i\in I} J_i\right) \middle| \, i \in I \right\},$$

can easily be shown to satisfy the conditions of Definition IV.33. Thus, there is a natural isomorphism $\mathfrak{Z}_{\mathrm{Fr}}(P) \cong \mathfrak{Z}(T^{\vee}P, J_P^{\vee})$ for each doctrine P by Theorem IV.35.

Free binary meet completion. Finally, we construct the *free binary meet completion* for doctrines, and observe that this is also a subgeometric completion. We begin by defining the free binary meet completion for preorders.

Definition IV.47. Let P be a preorder. Consider the set $\mathscr{P}_{\mathrm{fin}}(P) \setminus \emptyset$ of non-empty, finite subsets of P. We order $\mathscr{P}_{\mathrm{fin}}(P) \setminus \emptyset$ by setting

$$\{ x_1, x_2, \dots, x_n \} \leqslant \{ y_1, y_2, \dots, y_m \}$$

if and only if each y_i is greater than some x_j. We define $P^{\wedge}$ as the poset obtained by identifying two elements $\{ x_1, x_2, \dots, x_n \}$, $\{ y_1, y_2, \dots, y_m \}$ of $\mathscr{P}_{\mathrm{fin}}(P) \setminus \emptyset$ if

$$\{ x_1, x_2, \dots, x_n \} \leqslant \{ y_1, y_2, \dots, y_m \} \text{ and } \{ y_1, y_2, \dots, y_m \} \leqslant \{ x_1, x_2, \dots, x_n \}.$$

We denote the equivalence class of $\{ x_1, x_2, \dots, x_n \}$ by $[\![x_1, x_2, \dots, x_n]\!]$. Alternatively, $P^{\wedge}$ can be described as the poset of non-empty, finitely generated up-sets of P ordered by inclusion.

It is easily checked that, given two elements $[\![x_1, \dots, x_n]\!], [\![y_1, \dots, y_m]\!]$ of $P^{\wedge}$, their meet is given by

$$[\![x_1, \dots, x_n, y_1, \dots, y_m]\!],$$

and thus the poset $P^{\wedge}$ has all binary meets. The map $[\![-]\!]_P\colon P \to P^{\wedge}$ given by sending $x \in P$ to $[\![x]\!] \in P^{\wedge}$ is clearly monotone. Since every element $[\![x_1, \dots, x_n]\!] \in P^{\wedge}$ is the finite meet of the elements $[\![x_i]\!] \in P^{\wedge}$, we obtain the desired universal property: for each preorder P and any monotone map $a\colon P \to Q$, where Q has binary meets, there exists a unique monotone map $a^{\wedge}\colon P^{\wedge} \to Q$ that preserves binary meets such that the triangle

$$\begin{tikzcd} P \arrow[r, "{[\![-]\!]_P}"] \arrow[dr, "a"'] & P^{\wedge} \arrow[d, dashed, "a^{\wedge}"] \\ & Q \end{tikzcd}$$

commutes.

Thus, by the discussion above, there exists a left 2-adjoint $T^{\wedge}$ to the inclusion of **BMSLat**-valued doctrines into **Doc**, where **BMSLat** is the 2-category of binary-meet-semilattices, their homomorphisms, and natural transformations between these. Evidently, there exists a natural transformation $\mathrm{id}_{\mathbb{L}}^{\wedge}\colon T^{\wedge}\mathbb{L} \Rightarrow \mathbb{L}$, induced by the universal property of $T^{\wedge}$, which yields a $T^{\wedge}$ algebra $(\mathbb{L}, \mathrm{id}_{\mathbb{L}}^{\wedge})$ for each geometric doctrine $\mathbb{L} \in \mathbf{GeomDoc}$.

We denote by $J_P^\wedge$ the Grothendieck topology on $C \rtimes T^\wedge P$ generated by covering families of the form

$$\left\{ (c, [\![\, y \,]\!]) \xrightarrow{\mathrm{id}_c} (c, [\![\, x_1, x_2, \dots, x_n \,]\!]) \,\middle|\, y \in P(c),\ y \leqslant x_1, x_2, \dots, x_n \right\}.$$

There are few obstacles to concluding that the choice of topology $J_P^\wedge$ satisfies the conditions of Definition IV.33. Hence we obtain by Theorem IV.35 that there is a natural isomorphism

$$\mathfrak{Z}_{\mathrm{Fr}}(P) \cong \mathfrak{Z}(T^\wedge P, J_P^\wedge)$$

for every doctrine P.

Part B

Groupoidal Representations

Chapter V

Sheaves on a groupoid

Topoi as generalised spaces. Locales and topological spaces are generalised, respectively, by topoi and topoi with enough points[1]. But to what extent are spaces generalised by topoi? The representation results of Joyal and Tierney [68] and Butz and Moerdijk [17] express that, roughly speaking, a topos is a space whose points can possess non-trivial isomorphisms. In this regard, topoi can be likened to *orbifolds* from differential geometry.

Groupoids and their sheaves. The informal notion of a 'space with isomorphisms of points' is captured by the notion of a topological or localic groupoid. A topological/localic groupoid comes equipped with a natural notion of a topos of *equivariant sheaves*, or simply the topos of sheaves on a groupoid, which simultaneously generalises the topos of sheaves on a space and the topos of continuous actions by a topological group.

The representation results of [68] and [17] state that every topos (respectively, every topos with enough points) is equivalent to a topos of sheaves on localic (resp., topological) groupoid. We will review these representation results in Chapter VI and Chapter VII. But first, we recall in this chapter the definition of the topos of sheaves on a localic/topological groupoid, as well as its pertinent properties that will facilitate our later study.

Overview. We proceed as follows.

(A) For familiarity, we initially focus on the topological case. Section V.1 contains the definition and examples of topoi of sheaves on topological groupoids. In Section V.1.1, we study the behaviour of this topos when the action and topologies on the constituent spaces are modified.

(B) Secondly, we briefly recount in Section V.2 how our definitions adapt when topological groupoids are replaced by localic groupoids.

V.1 Sheaves on a topological groupoid

A *topological groupoid* is a groupoid internal to the category **Top** of topological spaces and continuous maps. That is, a topological groupoid $\mathbb{X}$ consists of a diagram

[1] Garner suggests the name *ionaid* (singular *ionad*) for the latter in [41].

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=3, "\mathrm{pr}_2"] \arrow[r, "m" description] \arrow[r, shift right=3, "\mathrm{pr}_1"'] & X_1 \arrow[loop below, "i"] \arrow[r, shift left=3, "t"] \arrow[r, shift right=3, "s"'] & X_0, \arrow[l, "e" description] \end{tikzcd} \tag{V.i}$$

in **Top** such that the equations

$$\begin{gathered} s \circ e = t \circ e = \mathrm{id}_{X_0}, \\ s \circ m = s \circ \mathrm{pr}_1, \quad t \circ m = t \circ \mathrm{pr}_2, \\ m \circ (\mathrm{id}_{X_1} \times_{X_0} m) = m \circ (m \times_{X_0} \mathrm{id}_{X_1}), \\ m \circ (\mathrm{id}_{X_1} \times_{X_0} e) = m \circ (e \times_{X_0} \mathrm{id}_{X_1}), \end{gathered}$$

expressing that (V.i) is an internal category (where s and t send an arrow to, respectively, its source and target, e sends an object to its identity morphism and m sends a pair of composable arrows to their composite), and

$$\begin{gathered} s \circ i = t, \quad t \circ i = s, \\ m \circ (\mathrm{id}_{X_1} \times_{X_0} i) = e \circ t, \\ m \circ (i \times_{X_0} \mathrm{id}_{X_1}) = e \circ s, \\ i \circ i = \mathrm{id}_{X_1}, \end{gathered}$$

expressing that i sends an arrow to its inverse, are all satisfied. Equivalently, a topological groupoid is a (small) groupoid in the usual sense and a choice of topologies for the set of objects and the set of arrows such that all the groupoid structure morphisms (i.e. the displayed arrows in (V.i)) are continuous with respect to these topologies.

Since we will mostly be concerned with the 'source' and 'target' maps s and t, we will often write $\mathbb{X} = (X_1 \rightrightarrows X_0)$ to denote the topological groupoid. As $s \circ i = t$, and i is a homeomorphism, s is open if and only if t is open. This allows us to simplify the definition of an open topological groupoid:

Definition V.1. A topological groupoid is said to be *open* if either s or t are open maps (and hence both are).

We will often restrict our focus to open topological groupoids. Of particular importance for us is the fact that in an open topological groupoid, the *orbit* $t(s^{-1}(U))$ of an open $U \subseteq X_0$, i.e. the closure of U under the action of X_1, is still open. The restriction to open topological groupoids is not prohibitive since every topological groupoid is *Morita equivalent* to an open one, in the sense that for every topological groupoid $\mathbb{X}$, its *topos of equivariant sheaves* (defined below) is equivalent to the sheaves on an open topological groupoid. This follows from [17].

Definition V.2. Given a topological groupoid $\mathbb{X}$, we can construct the topos of *equivariant sheaves* $\mathbf{Sh}(\mathbb{X})$. This is a construction that generalises simultaneously both topoi of sheaves on spaces and topoi of continuous group actions.

(i) Objects of $\mathbf{Sh}(\mathbb{X})$, called $\mathbb{X}$-*sheaves*, consist of triples (Y, q, β) where $q \colon Y \to X_0$ is a local homeomorphism and β is a continuous X_1-*action* on Y, by which mean a continuous map

$$\beta \colon Y \times_{X_0} X_1 \longrightarrow Y,$$

where $Y \times_{X_0} X_1$ is the pullback

$$\begin{tikzcd} Y \times_{X_0} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "q"] \\ X_1 \arrow[r, "s"] & X_0, \end{tikzcd}$$

satisfying the equations

$$\begin{aligned} \beta(\beta(y,g),h) &= \beta(y,m(g,h)), \\ q(\beta(y,g)) &= t(g), \\ \beta(y,e(q(y))) &= y. \end{aligned}$$

(ii) An arrow $(Y,q,\beta) \xrightarrow{f} (Y',q',\beta')$ of $\mathbf{Sh}(\mathbb{X})$ consists of a continuous map $f\colon Y \to Y'$ such that the diagram

$$\begin{tikzcd} Y \times_{X_0} X_1 \arrow[rr, "f \times_{X_0} \mathrm{id}_{X_1}"] \arrow[d, "\beta"] & & Y' \times_{X_0} X_1 \arrow[d, "\beta'"] \\ Y \arrow[rr, "f"] \arrow[dr, "q"'] & & Y' \arrow[dl, "q'"] \\ & X_0 & \end{tikzcd} \tag{V.ii}$$

commutes. The commutation of the bottom triangle of (V.ii) expresses that f is a a morphism of sheaves over X_0, while the commutation of the top square expresses that f respects the respective X_1-actions.

Examples V.3. That the topos of equivariant sheaves on a topological groupoid simultaneously generalises the topos of sheaves on a space and the topos of continuous actions by a topological group is clear by the following examples.

(i) Let X be a topological space. The diagram

$$\begin{tikzcd} X \arrow[r, shift left=2, "\mathrm{id}_X"] \arrow[r, "\mathrm{id}_X"] \arrow[r, shift right=2, "\mathrm{id}_X"'] & X \arrow[loop below, "\mathrm{id}_X"] \arrow[r, shift left=2, "\mathrm{id}_X"] \arrow[r, shift right=2, "\mathrm{id}_X"'] & X \arrow[l, "\mathrm{id}_X"'] \end{tikzcd}$$

is a topological groupoid whose topos of sheaves is the familiar topos of sheaves on a space $\mathbf{Sh}(X)$.

(ii) Let (G,e,m,i) be a topological group. The diagram

$$\begin{tikzcd} G \times G \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "m"] \arrow[r, shift right=2, "\mathrm{pr}_2"'] & G \arrow[loop below, "i"] \arrow[r, shift left=2, "!"] \arrow[r, shift right=2, "!"'] & 1 \arrow[l, "e"'] \end{tikzcd}$$

is a topological groupoid whose topos of sheaves is the topos $\mathbf{B}G$ of continuous group actions by G on discrete sets.

Descent data. The objects and morphisms of $\mathbf{Sh}(\mathbb{X})$ can be given a more compact definition in terms of *descent data* (the reasons for the nomenclature will become apparent in Section VI.2.1).

We first fix some notation. Recall that each continuous map $U \xrightarrow{h} V$ induces a geometric morphism $\mathbf{Sh}(h)\colon \mathbf{Sh}(U) \to \mathbf{Sh}(V)$, whose inverse image part we write as h^*. It sends a local homeomorphism $q\colon W \to V$ to its pullback along h

$$\begin{tikzcd}
h^*(W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, "q"] \\
U \arrow[r, "h"] & V,
\end{tikzcd}$$

and a morphism

$$\begin{tikzcd}
W' \arrow[rr, "g"] \arrow[dr] & & W \arrow[dl] \\
& V &
\end{tikzcd}$$

of $\mathbf{Sh}(Y)$ to the induced map

$$\begin{tikzcd}
h^*(W') \arrow[rr] \arrow[dr, dashed, "h^*(g)"] \arrow[ddr, bend right] & & W' \arrow[dr, "g"] \arrow[ddr, bend left] & \\
& h^*(W) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & W \arrow[d] \\
& U \arrow[rr, "h"] & & V.
\end{tikzcd}$$

A descent datum for $\mathbb{X}$ is a pair consisting of a local homeomorphism $q\colon Y \to X_0$ and a morphism

$$\begin{tikzcd}
s^*(Y) \arrow[rr, "\theta"] \arrow[dr] & & t^*(Y) \arrow[dl] \\
& X_1 &
\end{tikzcd}$$

such that $e^*(\theta) = \mathrm{id}_Y$ and $m^*(\theta) = \mathrm{pr}_2^*(\theta) \circ \mathrm{pr}_1^*(\theta)$. A morphism of descent data $(Y, \theta) \xrightarrow{f} (Y', \theta')$ is a commuting triangle

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "q"'] & & Y' \arrow[dl, "q'"] \\
& X_0 &
\end{tikzcd}$$

(i.e. a morphism $Y \xrightarrow{f} Y'$ in $\mathbf{Sh}(X_0)$) such that the square

$$\begin{tikzcd}
s^*(Y) \arrow[r, "\theta"] \arrow[d, "s^*(f)"'] & t^*(Y) \arrow[d, "t^*(f)"] \\
s^*(Y') \arrow[r, "\theta'"] & t^*(Y')
\end{tikzcd}$$

commutes.

That the two definitions of sheaves on $\mathbb{X}$ are equivalent is a matter of unravelling definitions. The notational difference arises because, for descent data, we keep track of the arrow $\alpha \in X_1$ once it has been applied to a point $y \in Y$. Indeed, given a X_1-action $\beta\colon Y \times_{X_0} X_1 \to Y$, the corresponding descent datum is the map θ_β that sends the pair $(y, \alpha) \in s^*(Y)$ to $(\beta(y, \alpha), \alpha) \in t^*(Y)$, while given descent datum $\theta\colon s^*(Y) \to t^*(Y)$ on Y, corresponding to the X_1-action β_θ, is the composite

$$Y \times_{X_0} X_1 = s^*(Y) \xrightarrow{\theta} t^*(Y) \longrightarrow Y.$$

For completeness, we explain the equivalence between X_1-actions and descent data in detail in Appendix B.

We will use actions and descent data interchangeably when discussing sheaves on a groupoid since, as we will also observe in Chapter VI, it can often be more convenient to work with one other over the other. For example, the following is most succinctly demonstrated using descent data.

Lemma V.4. *If $\mathbb{X} = (X_1 \rightrightarrows X_0)$ is an open topological groupoid, for any $\mathbb{X}$-sheaf (Y, q, β), the X_1-action*

$$\beta\colon Y \times_{X_0} X_1 \longrightarrow Y$$

is an open map.

Proof. By above (see also Appendix B), the action β is the composite

$$Y \times_{X_0} X_1 = s^*(Y) \xrightarrow{\theta} t^*(Y) \longrightarrow Y$$

for some descent datum θ on Y. The first factor $s^*(Y) \xrightarrow{\theta} t^*(Y)$, being a morphism of $\mathbf{Sh}(X_1)$, is an open continuous map (see [63, Lemma C1.3.2]). The later factor is also open since it is the pullback of the open map t in the square

$$\begin{tikzcd}
t^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X_1 \arrow[r, "t"] & X_0,
\end{tikzcd}$$

and open maps are stable under pullback. □

V.1.1 Forgetting topologies and actions

In this section we lay out the necessary facts regarding the sheaves on a topological groupoid that will be used in Chapter VII. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid, which can also be considered as a topological groupoid where both X_0 and X_1 have both been endowed with the discrete topology. We will write $\mathbb{X}_\delta^\delta = (X_1^\delta \rightrightarrows X_0^\delta)$ to emphasise this fact. Let τ_0 and τ_1 be a topologies on X_0 and X_1 respectively such that all the structure morphisms of $\mathbb{X}$ are continuous with respect to these topologies, i.e. $\mathbb{X}_{\tau_0}^{\tau_1} = (X_1^{\tau_1} \rightrightarrows X_0^{\tau_0})$ is a topological groupoid.

Definitions V.5. (i) As above, let $\mathbf{Sh}(\mathbb{X}_{\tau_0}^{\tau_1})$ and $\mathbf{Sh}(\mathbb{X}_\delta^\delta)$ denote the topoi of sheaves on the topological groupoids $\mathbb{X}_{\tau_0}^{\tau_1} = (X_1^{\tau_1} \rightrightarrows X_0^{\tau_0})$ and $\mathbb{X}_\delta^\delta = (X_1^\delta \rightrightarrows X_0^\delta)$.

(ii) By $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ we denote the category whose objects are local homeomorphisms $q\colon Y \to X_0^{\tau_0}$ equipped with a (not necessarily continuous) action $\beta\colon Y \times_{X_0} X_1 \to Y$, satisfying the same equations as in Definition V.2(i). Arrows $(Y, q, \beta) \to (Y', q', \beta')$ are continuous maps $f\colon Y \to Y'$ such that the diagram

$$\begin{tikzcd}
Y \times_{X_0} X_1 \arrow[rr, "f \times_{X_0} \mathrm{id}_{X_1}"] \arrow[d, "\beta"] & & Y' \times_{X_0} X_1 \arrow[d, "\beta'"] \\
Y \arrow[rr, "f"] \arrow[dr, "q"'] & & Y' \arrow[dl, "q'"] \\
& X_0 &
\end{tikzcd}$$

commutes, as in Definition V.2(ii).

Remark V.6. Note that $\mathbb{X}^{\delta}_{\tau_0} = (X_1^{\delta} \rightrightarrows X_0^{\tau_0})$ is *not* a topological groupoid (unless τ_0 is also the discrete topology). If $\mathbb{X}^{\delta}_{\tau_0} = (X_1^{\delta} \rightrightarrows X_0^{\tau_0})$ were a topological groupoid, then, for each $x \in X_0$, the singleton

$$\{\, x \,\} = e^{-1}(\{\, \mathrm{id}_x \,\})$$

would be an open subset of $X_0^{\tau_0}$. Despite this, the category $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ is still a topos, a consequence of [92, Theorem 2.5] and Lemma V.16 below.

We note that the topoi $\mathbf{Sh}(X_0^{\delta})$ and $\mathbf{Sh}(\mathbb{X}^{\delta}_{\delta})$ can be written in a more familiar manner. There are, of course, evident equivalences

$$\mathbf{Sh}(X_0^{\delta}) \simeq \mathbf{Sets}/X_0 \simeq \mathbf{Sets}^{X_0}, \;\; \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) \simeq \mathbf{Sets}^{\mathbb{X}}.$$

Section aims. The main focus of this section is to construct a commutative diagram of topoi and geometric morphisms

$$\begin{tikzcd}
\mathbf{Sh}(X_0^{\delta}) \arrow[r, "j"] \arrow[d, "v"] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d, "u^{\delta}"] \arrow[dr, "u"] & \\
& & \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \\
\mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) \arrow[r, "j'"] & \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \arrow[ur, "w"'] &
\end{tikzcd} \tag{V.iii}$$

such that the following are satisfied:

(i) j and u^{δ} are both localic surjections,

(ii) u is a localic surjection and, additionally, open if $\mathbb{X}^{\tau_1}_{\tau_0}$ is an open topological groupoid,

(iii) v is an open localic surjection,

(iv) j' is a surjection,

(v) w is a hyperconnected geometric morphism,

and the left-hand square is a pushout of topoi.

To construct the diagram (V.iii), we will make repeated use of [63, Theorem B2.4.6] to deduce that whenever a functor between topoi preserves finite limits and arbitrary colimits, it is the inverse image part of a geometric morphism.

Forgetting the action. We first note that the forgetful functors $U\colon \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \to \mathbf{Sh}(X_0^{\tau_0})$ and $U^\delta\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \to \mathbf{Sh}(X_0^{\tau_0})$, which forget the $X_1^{\tau_1}$-action (respectively, X_1^{δ}-action), create all colimits and finite limits. This is deduced since a colimit or finite limit in $\mathbf{Sh}(X_0^{\tau_0})$ of spaces with an $X_1^{\tau_1}$-action (resp., X_1^{δ}-action) can be given an obvious $X_1^{\tau_1}$-action (resp., X_1^{δ}-action) making it an object of $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ (resp., $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$).

Example V.7. We prove in more detail, as an example, that $U\colon \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \to \mathbf{Sh}(X_0^{\tau_0})$ preserves binary products, and remark that the other finite limits and arbitrary colimits follow just as easily.

Let (Y, q, β), (Y', q', β') be objects of $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$. The product of (Y, q) and (Y, q') in the topos $\mathbf{Sh}(X_0^{\tau_0})$ is given by the pullback of spaces

$$\begin{tikzcd} Y \times_{X_0} Y' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "q'"] \\ Y \arrow[r, "q"] & X^{\tau_0}. \end{tikzcd}$$

Let (y, y', α) be an element of $Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1}$, where $Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1}$ is the pullback

$$\begin{tikzcd} Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \times_{X_0} Y' \arrow[d] \\ X_1^{\tau_1} \arrow[r, "s"] & X^{\tau_0}. \end{tikzcd}$$

The definition $B(y, y', \alpha) = (\beta(y, \alpha), \beta'(y', \alpha))$ yields a $X_1^{\tau_1}$-action

$$B\colon Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1} \longrightarrow Y \times_{X_0} Y'.$$

The action B is continuous since β and β' are both continuous and the necessary equations on B are direct consequences of the equivalent equivalent equations for (Y, q, β) and (Y', q', β'). Hence, the triple $(Y \times_{X_0} Y', q \circ \mathrm{pr}_1, B)$ is an object of $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$.

The projections $Y \times_{X_0} Y' \xrightarrow{\mathrm{pr}_1} Y$ and $Y \times_{X_0} Y' \xrightarrow{\mathrm{pr}_2} Y'$ define morphisms in $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ since, by a simple diagram chase, both the diagrams

$$\begin{tikzcd} Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1} \arrow[rr, "\mathrm{pr}_1 \times_{X_0} \mathrm{id}_{X_1}"] \arrow[d, "B"] & & Y \times_{X_0} X_1^{\tau_1} \arrow[d, "\beta"] \\ Y \times_{X_0} Y' \arrow[rr, "\mathrm{pr}_1"] \arrow[dr, "q \circ \mathrm{pr}_1"'] & & Y \arrow[dl, "q"] \\ & X_0^{\tau_0}, & \end{tikzcd} \qquad \begin{tikzcd} Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1} \arrow[rr, "\mathrm{pr}_2 \times_{X_0} \mathrm{id}_{X_1}"] \arrow[d, "B"] & & Y' \times_{X_0} X_1^{\tau_1} \arrow[d, "\beta'"] \\ Y \times_{X_0} Y' \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "q \circ \mathrm{pr}_1 = q' \circ \mathrm{pr}_2"'] & & Y' \arrow[dl, "q'"] \\ & X_0^{\tau_0}. & \end{tikzcd}$$

commute.

Finally, we can demonstrate that $(Y \times_{X_0} Y', q \circ \mathrm{pr}_1, B)$ satisfies the universal property of the product. Let (Z, p, γ) be a $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaf with morphisms f and g to (Y, q, β) and

(Y', q', β') respectively. Then there is a unique commuting continuous map $Z \xrightarrow{h} Y \times_{X_0} Y'$ induced by the pullback

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_{X_0} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "q'"] \\
& Y \arrow[r, "q"] & X_0^{\tau_0}
\end{tikzcd}$$

that sends $z \in Z$ to $(f(z), g(z)) \in Y \times_{X_0} Y'$. Thus, $(Y \times_{X_0} Y', q \circ \mathrm{pr}_1, B)$ is the product of (Y, q, β) and (Y', q', β') in the category $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ if h makes the diagram

$$\begin{tikzcd}
Z \times_{X_0} X_1^{\tau_1} \arrow[rr, "h \times_{X_0} \mathrm{id}_{X_1}"] \arrow[d, "\gamma"] & & Y \times_{X_0} Y' \times_{X_0} X_1^{\tau_1} \arrow[d, "B"] \\
Z \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \times_{X_0} Y' \arrow[dl, "q \circ \mathrm{pr}_1"] \\
& X_0^{\tau_0} &
\end{tikzcd}$$

commute. This is easily checked by another diagram chase, and so we have demonstrated that $U \colon \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \to \mathbf{Sh}(X_0^{\tau_0})$ preserves binary products.

Since U (respectively, U^δ) preserves finite limits and all colimits, by [63, Theorem B2.4.6], it is the inverse image of a geometric morphism $u \colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ (resp., $u^\delta \colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$) between topoi.

Lemma V.8. *The geometric morphisms $u \colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ and $u^\delta \colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ are both localic surjections.*

Proof. As U (respectively, U^δ) is clearly a faithful functor whose codomain $\mathbf{Sh}(X_0^{\tau_0})$ is a localic topos, u (resp., u^δ) is a surjective localic geometric morphism. □

Lemma V.9 (Proposition 4.4 [92])**.** *If $\mathbb{X}^{\tau_1}_{\tau_0}$ is an open topological groupoid, then the geometric morphism $u \colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ is additionally open.*

Remark V.10. The functors U and U^δ above reflect jointly epimorphic families and monomorphisms. As shown in [79, Proposition II.6.6], a family of morphisms in $\mathbf{Sh}(X_0^{\tau_0})$ is jointly epimorphic if and only if they are jointly surjective, and hence so too in $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ and $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$. Also, a morphism in $\mathbf{Sh}(X_0^{\tau_0})$ is a monomorphism if and only if it is injective, and hence so too in $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ and $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$.

Let $V \colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) \to \mathbf{Sh}(X_0^{\delta})$ denote the analogous functor that forgets the X_1^δ-action. By an identical analysis to the above, we conclude the following.

Lemma V.11. *The functor V is the inverse image functor of a geometric morphism*

$$v \colon \mathbf{Sh}(X_0^{\delta}) \longrightarrow \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta})$$

that is open, localic and surjective.

Subsheaves. Given an $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaf (Y, q, β), the subobjects of (Y, q, β) are easy to describe. By Remark V.10, a morphism $(Y, q, \beta) \xrightarrow{f} (Y', q', \beta')$ of $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaves is a monomorphism in $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ if and only if $Y \xrightarrow{f} Y'$ is a monomorphism in $\mathbf{Sh}(X_0)$, i.e. f is the inclusion of an open subspace. The requirement that f makes the diagram (V.ii) commute is equivalent to the following.

Definition V.12. Given an $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaf (Y, q, β), a subspace $Y' \subseteq Y$ is said to be *stable*[2] if Y' is closed under the X_1-action β on Y, by which we mean that if $y \in Y' \subseteq Y$ then $\beta(y, \alpha) \in Y' \subseteq Y$ too, for any suitable $\alpha \in X_1$.

Lemma V.13. *Therefore, the subobjects of a $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaf (Y, q, β) can be identified with the $\mathbb{X}^{\tau_1}_{\tau_0}$-sheaves $(U, q \circ i, \beta|_U)$, where $i\colon U \hookrightarrow Y$ is the inclusion of an open and stable subspace.*

Forgetting the topology on arrows. The topos $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ is evidently a full subcategory of $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$. Let $W\colon \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ be the inclusion functor. Clearly, there is a commuting triangle of functors

$$\begin{tikzcd}
\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \arrow[r, "W"] \arrow[dr, "U"'] & \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \arrow[d, "U^\delta"] \\
& \mathbf{Sh}(X_0^{\tau_0}).
\end{tikzcd}$$

Hence, as U preserves finite limits and arbitrary colimits while U^δ reflects them, W also preserves finite limits and arbitrary colimits. Therefore, W is the inverse image of a geometric morphism $w\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$.

Proposition V.14. *The geometric morphism $w\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ is hyperconnected.*

Proof. A hyperconnected geometric morphism is one whose inverse image functor is full and faithful and whose image is closed under subobjects (see [63, Proposition A4.6.6]). The functor $W\colon \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ is already full and faithful by definition.

Let (Y, q, β) be a $\mathbb{X}^{\delta}_{\tau_0}$-space whose X_1^{δ}-action β becomes a continuous map

$$\beta\colon Y \times_{X_0} X_1^{\tau_1} \longrightarrow Y$$

when X_1 is endowed with the topology τ_1. If Z is a subobject of Y in the topos $\mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$, then Z is an open subspace of Y whose X_1^{δ}-action is the restriction of β to the subset

$$Z \times_{X_0} X_1 \subseteq Y \times_{X_0} X_1.$$

Since $\beta|^{-1}_{Z \times_{X_0} X_1}(U) = \beta^{-1}(U) \cap (Z \times_{X_0} X_1)$, for each open subset $U \subseteq Z$, and as β is continuous for the topology on $Y \times_{X_0} X_1^{\tau_1}$, so too is $\beta|_{Z \times_{X_0} X_1}\colon Z \times_{X_0} X_1^{\tau_1} \to Z$. Thus, the image of W is closed under subobjects. ◻

[2]Note that we are following the terminology of [5], [36], [37], where the term ‘stable’ was used to reduce confusion with closed subspaces.

Remark V.15. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a groupoid that becomes a topological groupoid when X_0 is endowed with the topology τ_0 and X_1 is endowed with τ_1. The construction of the hyperconnected morphism $w\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ relied only on the fact that the discrete topology δ contains the topology τ_1. Indeed, for any other topology τ_1' on X_1, containing τ_1, such that

$$\mathbb{X}^{\tau_1'}_{\tau_0} = \left(X_1^{\tau_1'} \rightrightarrows X_0^{\tau_0}\right)$$

is also a topological groupoid, then there is a hyperconnected geometric morphism

$$\mathbf{Sh}\left(\mathbb{X}^{\tau_1'}_{\tau_0}\right) \longrightarrow \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$$

whose inverse image is the inclusion of $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$ into $\mathbf{Sh}\left(\mathbb{X}^{\tau_1'}_{\tau_0}\right)$.

Forgetting the topology on objects. The identity $\mathrm{id}_{X_0}\colon X_0^{\delta} \to X_0^{\tau_0}$ is a surjective continuous map of topological spaces and so induces (see [79, §II.9 & §IX.4]) a surjective localic geometric morphism

$$j\colon \mathbf{Sh}(X_0^{\delta}) \longrightarrow \mathbf{Sh}(X_0^{\tau_0}).$$

The inverse image of j is the functor $J\colon \mathbf{Sh}(X_0^{\tau_0}) \to \mathbf{Sh}(X_0^{\delta})$ that sends a local homeomorphism $q\colon Y \to X_0^{\tau_0}$ the pullback of q along $\mathrm{id}_{X_0}\colon X_0^{\delta} \to X_0^{\tau_0}$. In other words, J is the functor that forgets the topology on Y. We denote $J(Y)$ by Y^{δ}.

There is a similar forgetful functor $J'\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta})$ that sends a $\mathbb{X}^{\delta}_{\tau_0}$-space (Y, q, β) to (Y^{δ}, q, β). Clearly, there is a commutative square

$$\begin{array}{ccc} \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) & \xrightarrow{U^{\delta}} & \mathbf{Sh}(X_0^{\tau_0}) \\ {\scriptstyle J'}\downarrow & & \downarrow{\scriptstyle J} \\ \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) & \xrightarrow{V} & \mathbf{Sh}(X_0^{\delta}). \end{array}$$

As $J \circ U^{\delta}$ preserves finite limits and arbitrary colimits and V reflects them, J' preserves finite limits and arbitrary colimits too. Therefore, J' is the inverse image of a geometric morphism $j'\colon \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) \to \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0})$ which makes the square

$$\begin{array}{ccc} \mathbf{Sh}(X_0^{\delta}) & \xrightarrow{j} & \mathbf{Sh}(X_0^{\tau_0}) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u^{\delta}} \\ \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) & \xrightarrow{j'} & \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \end{array}$$

commute. Moreover, since j, u^{δ} and v are surjective geometric morphisms, so too is j' (since surjective geometric morphisms are the left class in an orthogonal factorisation system, see Theorem A4.2.10 [63]).

Lemma V.16. *The square*

$$\begin{array}{ccc} \mathbf{Sh}(X_0^{\delta}) & \xrightarrow{j} & \mathbf{Sh}(X_0^{\tau_0}) \\ {\scriptstyle v}\downarrow & \lrcorner & \downarrow{\scriptstyle u^{\delta}} \\ \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) & \xrightarrow{j'} & \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \end{array}$$

is a (bi)pushout in the category **Topos** *of Grothendieck topoi and geometric morphisms.*

Proof. By [92, Theorem 2.5], the bipushout of the diagram

$$\begin{tikzcd} \mathbf{Sh}(X_0^\delta) \arrow[r, "j"] \arrow[d, "v"'] & \mathbf{Sh}(X_0^{\tau_0}) \\ \mathbf{Sh}(\mathbb{X}_\delta^\delta) & \end{tikzcd}$$

in **Topos** is computed as the bipullback (see [76, Example 15]) of the inverse image functors

$$\begin{tikzcd} & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d, "J"'] \\ \mathbf{Sh}(\mathbb{X}_\delta^\delta) \arrow[r, "V"] & \mathbf{Sh}(X_0^\delta) \end{tikzcd} \tag{V.iv}$$

in $\mathfrak{CAT}$, the category of (large) categories.

It is then easy to see that the commuting square

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta) \arrow[r, "U^\delta"] \arrow[d, "J'"] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d, "J"] \\ \mathbf{Sh}(\mathbb{X}_\delta^\delta) \arrow[r, "V"] & \mathbf{Sh}(X_0^\delta) \end{tikzcd} \tag{V.v}$$

is said bipullback. Given a bicone $F\colon A \to \mathbf{Sh}(X_0^{\tau_0})$, $G\colon A \to \mathbf{Sh}(\mathbb{X}_\delta^\delta)$ of the cospan (V.iv), i.e. $J \circ F \cong V \circ G$, there is a unique (up to isomorphism) functor such that the diagram

$$\begin{tikzcd} A \arrow[dr, dashed] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "G"'] & & \\ & \mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta) \arrow[r, "U^\delta"] \arrow[d, "J'"'] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d, "J"] \\ & \mathbf{Sh}(\mathbb{X}_\delta^\delta) \arrow[r, "V"] & \mathbf{Sh}(X_0^\delta). \end{tikzcd}$$

commutes up to isomorphism. The functor $A \dashrightarrow \mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta)$ is constructed as follows.

(i) For an object $a \in A$, $F(a)$ is a local homeomorphism $Y \to X_0^{\tau_0}$. Since

$$Y^\delta = J \circ F(a) \cong V \circ G(a),$$

the set Y^δ can be endowed with the (non-continuous) X_1^δ-action given by $G(a)$, thus defining $Y \to X_0^{\tau_0}$ as an object of $\mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta)$.

(ii) Each arrow g of A is sent by F to a continuous map $f\colon Y \to Y'$ for which the triangle

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr] & Y' \arrow[d] \\ & X_0^{\tau_0} \end{tikzcd}$$

commutes. Again using that $f = V \circ G(g) \cong J \circ F(g)$, we deduce that f is also equivariant with respect to the imposed X_1^δ-actions on Y^δ and Y'^δ. Thus, f also defines an arrow of $\mathbf{Sh}(\mathbb{X}^\delta_{\tau_0})$.

It is clear by definition that the diagram (V.v) commutes up to natural isomorphism.

It remains to show that these natural isomorphisms also satisfy the universal property required by the bipullback. However, we can elide these details since, as the perceptive reader will notice, by applying the same reasoning as above, $\mathbf{Sh}(\mathbb{X}^\delta_{\tau_0})$ is also the 1-pullback of the cospan (V.iv). By [67], we know that the bipullback and the 1-pullback are equivalent since the functor $V \colon \mathbf{Sh}(\mathbb{X}^\delta_\delta) \to \mathbf{Sh}(X_0^\delta)$ is easily observed to satisfy the *invertible-path lifting property*. If f is an X_1^δ-equivariant map of sets over X_0^δ such that $V(f)$ has an inverse in $\mathbf{Sh}(X_0^\delta)$, then this inverse must also be X_1^δ-equivariant, and so defines an inverse for f in $\mathbf{Sh}(\mathbb{X}^\delta_\delta)$. □

This completes the construction of the diagram (V.iii) and the demonstration of its required properties.

V.2 Sheaves on a localic groupoid

The theory of sheaves on topological groupoids can be repeated for localic groupoids.

Definition V.17. A *localic groupoid* $\mathbb{Y}$,

$$\begin{tikzcd} Y_1 \times_{Y_0} Y_1 \arrow[r, "m"] & Y_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & Y_0, \arrow[l, "e"' description] \end{tikzcd}$$

is a groupoid internal to the category **Loc**. By replacing each instance of 'topological space' in Definition V.2 with locale and each instance of 'continuous map' with locale morphism, we obtain a topos $\mathbf{Sh}(\mathbb{Y})$, the *topos of sheaves on* $\mathbb{Y}$.

Remark V.18. As explained in [92, §5.3], we can re-express equations in locale theory in the more familiar notation of point-set topology, provided a 'point' $y \in Y$ is taken to mean a 'generalised point' of Y, i.e. an arbitrary locale morphism $y \colon U \to Y$. To translate a 'point-set' argument back to a concrete one, each instance of $y \in Y$ should be replaced by a generic locale morphism $y \colon U \to Y$, and the notation $f(y)$ for some map $f \colon Y \to X$ is translated as the composite $f \circ y \colon U \to Y \to X$.

For example, given a $\mathbb{Y}$-sheaf (Z, q, β), the point-set equation

$$\forall z \in Z \;\; \beta(z, e(q(z))) = z$$

satisfied by (Z, q, β) expresses the commutativity of the triangle

$$\begin{tikzcd} Z \arrow[r, "(\mathrm{id}_Z, e \circ q)"] \arrow[dr, equal] & Z \times_{Y_0} Y_1 \arrow[d, "\beta"] \\ & Z, \end{tikzcd}$$

where $(\mathrm{id}_Z, e \circ q)$ denotes the universally induced map

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "\mathrm{id}_Z"] \arrow[rd, dashed, "(\mathrm{id}_Z, e\circ q)"] \arrow[dd, "q"'] & & \\
 & Z \times_{Y_0} Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "q"] \\
Y_0 \arrow[r, "e"] & Y_1 \arrow[r, "s"] & Y_0.
\end{tikzcd}$$

Lemma V.8 and Lemma V.9 also apply to localic groupoids. That is, for each localic groupoid $\mathbb{Y}$, there is a surjective geometric morphism $u\colon \mathbf{Sh}(Y_0) \to \mathbf{Sh}(\mathbb{Y})$ whose inverse image is the functor that forgets the Y_1-action on sheaves, and moreover u is open if and only if $\mathbb{Y}$ is an open localic groupoid.

To conclude this chapter, we discuss obtaining localic groupoids from topological groupoids and vice versa, and compare their topoi of sheaves.

From topological groupoids to localic groupoids. Since the functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Loc}$ that sends a topological space to its locale of opens does not, in general, preserve limits, if $\mathbb{X}$ is a topological groupoid

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, "m"] & X_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & X_0, \arrow[l, "e"']
\end{tikzcd}$$

there is no reason for $\mathcal{O}(\mathbb{X})$, i.e. the diagram of locales and locale morphisms

$$\begin{tikzcd}
\mathcal{O}(X_1 \times_{X_0} X_1) \arrow[r, "\mathcal{O}(m)"] & \mathcal{O}(X_1) \arrow[loop below, "\mathcal{O}(i)"] \arrow[r, shift left=2, "\mathcal{O}(t)"] \arrow[r, shift right=2, "\mathcal{O}(s)"'] & \mathcal{O}(X_0), \arrow[l, "\mathcal{O}(e)"']
\end{tikzcd}$$

to define a localic groupoid since $\mathcal{O}(X_1 \times_{X_0} X_1) \ncong \mathcal{O}(X_1) \times_{\mathcal{O}(X_0)} \mathcal{O}(X_1)$.

However, when $\mathcal{O}(\mathbb{X})$ does define a localic groupoid, the topoi $\mathbf{Sh}(\mathbb{X})$ and $\mathbf{Sh}(\mathcal{O}(X))$ are equivalent. This follows from the fact that, for a local homeomorphism between locales $q\colon W \to V$, if V is spatial then W is spatial too, and that local homeomorphisms are stable under pullback (see [63, Lemma C1.3.2]). Thus, the topological $\mathbb{X}$-sheaves coincide with the localic $\mathcal{O}(\mathbb{X})$-sheaves.

From localic groupoids to topological groupoids. Conversely, the functor

$$\mathrm{Pt}\colon \mathbf{Loc} \longrightarrow \mathbf{Top}$$

that sends a locale to its space of points (see [60, §II.1]), being a right adjoint, preserves all limits. Thus, if $\mathbb{Y}$ is a localic groupoid

$$\begin{tikzcd}
Y_1 \times_{Y_0} Y_1 \arrow[r, "m"] & Y_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & Y_0, \arrow[l, "e"']
\end{tikzcd}$$

then $\mathrm{Pt}(\mathbb{Y})$ is a topological groupoid, where $\mathrm{Pt}(\mathbb{Y})$ denotes the diagram

$$\begin{tikzcd} \mathrm{Pt}(Y_1) \times_{\mathrm{Pt}(Y_0)} \mathrm{Pt}(Y_1) \cong \mathrm{Pt}(Y_1 \times_{Y_0} Y_1) \arrow[r, "\mathrm{Pt}(m)"] & \mathrm{Pt}(Y_1) \arrow[loop below, "\mathrm{Pt}(i)"] \arrow[r, shift left=2, "\mathrm{Pt}(t)"] \arrow[r, shift right=2, "\mathrm{Pt}(s)"'] & \mathrm{Pt}(Y_0). \arrow[l, "\mathrm{Pt}(e)"' description] \end{tikzcd}$$

In contrast to the topoi $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathcal{O}(\mathbb{X}))$ above, the topoi $\mathbf{Sh}(\mathbb{Y})$ and $\mathbf{Sh}(\mathrm{Pt}(\mathbb{Y}))$ can be very different. For example, $\mathbb{Y}$ could be chosen as the localic groupoid

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}_Y"] & Y \arrow[loop below, "\mathrm{id}_Y"] \arrow[r, shift left=2, "\mathrm{id}_Y"] \arrow[r, shift right=2, "\mathrm{id}_Y"'] & Y, \arrow[l, "\mathrm{id}_Y" description] \end{tikzcd}$$

where Y is a non-trivial locale without points (see Example V.19 for an example), in which case $\mathbf{Sh}(\mathrm{Pt}(\mathbb{Y}))$ is the trivial topos $\mathbf{0}_{\mathbf{Topos}}$ and therefore

$$\mathbf{Sh}(Y) \simeq \mathbf{Sh}(\mathbb{Y}) \not\simeq \mathbf{Sh}(\mathrm{Pt}(\mathbb{Y})).$$

However, if the locale of objects Y_0, the locale of arrows Y_1, and the locale of composable arrows $Y_1 \times_{Y_0} Y_1$ in $\mathbb{Y}$ are all spatial, then $\mathcal{O}\mathrm{Pt}(\mathbb{Y}) \cong \mathbb{Y}$ and so there is an equivalence of topoi $\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathcal{O}\mathrm{Pt}(\mathbb{Y})) \simeq \mathbf{Sh}(\mathrm{Pt}(\mathbb{Y}))$.

Example V.19 (Partial surjections from $\mathbb{N}$ to X)**.** Let X be an uncountable set, and let $\mathbb{T}_{\mathbb{N} \rightharpoondown\!\!\!\!\rightarrow X}$ be the propositional geometric theory

(i) with a basic proposition $[f(n) = x]$, for each $n \in \mathbb{N}$ and $x \in X$,

(ii) and the axioms, for every $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$,

$$[f(n) = x] \wedge [f(n) = y] \vdash \bot,$$
$$\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x].$$

A $\mathbf{2}$-valued model of $\mathbb{T}_{\mathbb{N} \rightharpoondown\!\!\!\!\rightarrow X}$ corresponds to a partial surjection $\mathbb{N} \rightharpoondown\!\!\!\!\rightarrow X$, which cannot exist as X is uncountable, and so there are no $\mathbf{2}$-valued models of $\mathbb{T}_{\mathbb{N} \rightharpoondown\!\!\!\!\rightarrow X}$.

Nonetheless, the classifying locale of $\mathbb{T}_{\mathbb{N} \rightharpoondown\!\!\!\!\rightarrow X}$ is non-trivial (see [63, Example C1.2.8]), and thus is an example of a non-trivial locale without any points. Informally, this expresses that, from a localic perspective, even uncountable sets are subquotients of $\mathbb{N}$. This will prove important in Chapter VI.

Chapter VI

A localic representing groupoid

Localic representation of predicate theories. Joyal and Tierney famously proved in [68] that every topos $\mathcal{E}$ (and hence every geometric theory) is *represented* by a localic groupoid $\mathbb{Y}$, by which we mean that there is an equivalence $\mathcal{E} \simeq \mathbf{Sh}(\mathbb{Y})$. This expresses that the topos $\mathcal{E}$ can be thought of as a 'space', in the pointfree sense, equipped with further 'isomorphisms of the points'.

The original paper [68] presents a general method of constructing a representing localic groupoid for a topos $\mathcal{E}$ from any *open cover* $\mathcal{F} \twoheadrightarrow \mathcal{E}$ (see Definition VI.7). However, potentially because of the level of abstraction involved, there is some confusion as to how to construct a representing localic groupoid in concrete cases [103], [109], [117].

Our goals. The purpose of this chapter is twofold.

(A) We provide a review of the Joyal-Tierney construction in order to compare how the theory for localic groupoids differs from the representation of topoi by topological groupoids discussed in Chapter VII. This is performed in Section VI.2.

(B) Our ultimate aim is to write down an explicit description of a representing localic groupoid for the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$. Since every topos is the classifying topos of some theory, this provides a description of a representing localic groupoid for any topos. Our description, provided in Section VI.3, will prove familiar when we later recall the representing topological groupoids studied in [5], [17], [36], [37].

This chapter is adapted from joint work with Graham Manuell [88].

VI.1 Reasoning using points

Prior to embarking on the a description of the Joyal-Tierney result, we remark that, just as for locales (see [92, §5.3] or Remark V.18), we can also use generalised points of topoi, i.e. arbitrary geometric morphisms $f\colon \mathcal{E}' \to \mathcal{E}$, in order to reason about them as though they were spaces (see [123]) – though in this case we must also consider morphisms of points since topoi exist at a higher categorical level than locales.

This is especially useful when combined with the theory of classifying topoi, since we can define a geometric morphism $g\colon \mathcal{E}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}'}$ by describing how g acts on a (generalised) point $\mathcal{F} \to \mathcal{E}_{\mathbb{T}}$ and morphisms of these points. That is to say, we can define g by describing how it transforms a $\mathbb{T}$-model (in $\mathcal{F}$) into a $\mathbb{T}'$-model and a $\mathbb{T}$-model homomorphism into a $\mathbb{T}'$-model homomorphism.

For example, given a geometric theory $\mathbb{T}$ over a signature Σ with N sorts, the associated localic geometric morphism

$$C_{\pi_{F^{\mathbb{T}}}} : \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(F^{\mathbb{T}}) \longrightarrow \mathbf{Sets}^{\mathbf{Con}_N} \simeq \mathcal{E}_{N \cdot \mathbb{O}}$$

deduced from Proposition III.42 sends a $\mathbb{T}$-model in a topos $\mathcal{F}$ to the N objects of its underlying sorts and a $\mathbb{T}$-model homomorphism to the N underlying functions between these objects.

This perspective lends itself well to the problem of determining the geometric theory classified by certain (bi)limits of other classifying topoi, using the method described in [123, Proposition 8.43].

Examples VI.1. Let us consider some examples of how to compute limits with this approach.

(i) Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories. The data of an $\mathcal{F}$-point of the product topos $\mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\mathbb{T}'}$ is a pair of geometric morphisms $\mathcal{F} \to \mathcal{E}_{\mathbb{T}}$ and $\mathcal{F} \to \mathcal{E}_{\mathbb{T}'}$, i.e. a pair of a $\mathbb{T}$-model and a $\mathbb{T}'$-model in $\mathcal{F}$. Thus, we conclude that the product topos $\mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\mathbb{T}'}$ classifies the theory given by a copy of $\mathbb{T}$ and a copy of $\mathbb{T}'$ (over separate sorts).

(ii) Let $\mathbb{T}_1, \mathbb{T}_2$ be *localic expansions* (see [22, §7.1] or Definition III.44) of a theory $\mathbb{T}_3$, i.e. all three theories share the same sorts, but the theories $\mathbb{T}_1$ and $\mathbb{T}_2$ add new symbols and new axioms to $\mathbb{T}_3$. Let $e^{\mathbb{T}_1}_{\mathbb{T}_3} : \mathcal{E}_{\mathbb{T}_1} \to \mathcal{E}_{\mathbb{T}_3}$ be the localic geometric morphism induced by Corollary III.45. It is the geometric morphism that acts on (generalised) points by sending a $\mathbb{T}_1$-model to its $\mathbb{T}_3$-*reduct*, i.e. the $\mathbb{T}_3$-model obtained when we forget the extra structure added by $\mathbb{T}_1$, and which sends a $\mathbb{T}_1$-model homomorphism to its underlying homomorphism on the $\mathbb{T}_3$-reducts. Similarly, the morphism $e^{\mathbb{T}_2}_{\mathbb{T}_3} : \mathcal{E}_{\mathbb{T}_2} \to \mathcal{E}_{\mathbb{T}_3}$ sends a $\mathbb{T}_2$-model to its $\mathbb{T}_3$-reduct.

An $\mathcal{F}$-point of the (bi)pullback

$$\begin{tikzcd} \mathcal{E}_{\mathbb{T}_1} \times_{\mathcal{E}_{\mathbb{T}_3}} \mathcal{E}_{\mathbb{T}_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_{\mathbb{T}_2} \arrow[d, "e^{\mathbb{T}_2}_{\mathbb{T}_3}"] \\ \mathcal{E}_{\mathbb{T}_1} \arrow[r, "e^{\mathbb{T}_1}_{\mathbb{T}_3}"'] & \mathcal{E}_{\mathbb{T}_3} \end{tikzcd}$$

consists of the data of a pair of $\mathcal{F}$-points $M : \mathcal{F} \to \mathcal{E}_{\mathbb{T}_1}$ and $N : \mathcal{F} \to \mathcal{E}_{\mathbb{T}_2}$ and an isomorphism

$$e^{\mathbb{T}_1}_{\mathbb{T}_3} \circ M \cong e^{\mathbb{T}_2}_{\mathbb{T}_3} \circ N.$$

Therefore, the (bi)pullback topos $\mathcal{E}_{\mathbb{T}_1} \times_{\mathcal{E}_{\mathbb{T}_3}} \mathcal{E}_{\mathbb{T}_2}$ classifies the theory whose models are a pair of a $\mathbb{T}_1$-model and a $\mathbb{T}_2$-model whose $\mathbb{T}_3$-reducts are isomorphic.

Remark VI.2. Some readers may wonder how our theory is impacted when we vary the specific notion of 2-limit we consider. Ultimately, as classifying topoi are defined up to equivalence, this won't be of importance. We will focus on comparing, for a geometric theory $\mathbb{T}$, the various notions of 'pullback' for the diagram

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] \\ \mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] & \mathcal{E}_{\mathbb{T}}. \end{tikzcd}$$

Evidently, the 1-pullback is given simply by $\mathcal{E}_{\mathbb{T}}$.

When calculating the bipullback as in Examples VI.1 above, we are implicitly taking the *iso-comma object* of the cospan. This is the topos $\mathcal{E}$ that is universal with respect to the data of projections $r, u \colon \mathcal{E} \rightrightarrows \mathcal{E}_{\mathbb{T}}$ and an isomorphism

$$\begin{tikzcd} \mathcal{E} \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] \\ \mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"'] & \mathcal{E}_{\mathbb{T}}. \end{tikzcd}$$

As in Examples VI.1(ii), we recognise that $\mathcal{E}$ classifies the theory of $\mathbb{T}$-model isomorphisms. We denote this theory by $\mathbb{T}_{\cong}$. An explicit axiomatisation of this theory is given in Definition VI.14 below.

Subtle changes to the notion of 2-pullback we take can change the specific presentation for the theory classified by the topos. For example, if we instead considered the *pseudo-pullback*, i.e. the topos $\mathcal{E}'$ that is universal with respect to the data

$$\begin{tikzcd} \mathcal{E}' \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, dashed] \arrow[r, phantom, shift right=4, "\cong"] \arrow[d, phantom, shift left=4, "\cong"] & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] \\ \mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"'] & \mathcal{E}_{\mathbb{T}}, \end{tikzcd}$$

we see that $\mathcal{E}'$ classifies the theory $\mathbb{T}_{\cong,\cong}$ whose models are triples of $\mathbb{T}$-models and a pair of isomorphisms between these.

However, such care will not be necessary. Recall from [76, Example 15] that although the topoi $\mathcal{E}_{\mathbb{T}_{\cong}}$ and $\mathcal{E}_{\mathbb{T}_{\cong,\cong}}$ are not isomorphic as categories, they are equivalent. In fact, the iso-comma object $\mathcal{E}_{\mathbb{T}_{\cong}}$, the pseudo-pullback $\mathcal{E}_{\mathbb{T}_{\cong,\cong}}$ and the the (1-)pullback $\mathcal{E}_{\mathbb{T}}$ are all equivalent by an application of [67]. We sidestep these issues by only working up to equivalence and referring to bipullbacks. Consequently, the theories $\mathbb{T}$, $\mathbb{T}_{\cong}$ and $\mathbb{T}_{\cong,\cong}$ are all Morita equivalent.

VI.2 Overview of the Joyal-Tierney theorem

We now give an overview of the Joyal-Tierney result from [68]. A description of the representing localic groupoid of the classifying topos $\mathcal{E}_{\mathbb{T}}$ constructed via the Joyal-Tierney method is provided in Section VI.3.1.

This section can be summarised as follows.

- In Section VI.2.1, we recall the theory of descent exposited in [68]. Given a geometric morphism $f \colon \mathcal{F} \to \mathcal{E}$, this is a way to study objects of $\mathcal{E}$ by equipping objects of $\mathcal{F}$ with additional data. This data forms a topos $\mathbf{Desc}_f(\mathcal{F}_\bullet)$. If the geometric morphism $f\colon \mathcal{F} \twoheadrightarrow \mathcal{E}$ is an open surjection, then there is an equivalence of topoi $\mathbf{Desc}_f(\mathcal{F}_\bullet) \simeq \mathcal{E}$.

– In Section VI.2.2 we note that $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ is naturally represented by a localic groupoid whenever $\mathcal{F}$ is a localic topos. Therefore, one can obtain a representation of $\mathcal{E}$ by a localic groupoid from an open surjection $\mathcal{F} \twoheadrightarrow \mathcal{E}$ whose domain is localic (called an *open cover*).

– Finally, in Section VI.2.3 we construct an open cover of a topos $\mathcal{E}$ from a geometric theory classified by $\mathcal{E}$, and hence conclude the Joyal-Tierney theorem that every topos is the topos of sheaves on some localic groupoid.

VI.2.1 Descent theory

In order to prove their representation theorem, Joyal and Tierney developed in [68] a *descent theory* for topoi. We will treat descent theory as a 'black box', recalling below the necessary facts we will use in our exposition. For details, the reader is directed to [68, §VIII] and [63, §B1.5 and §C5.1].

Let C be a cartesian 1-category. Recall that the pullback of an arrow $c \xrightarrow{f} d$ along itself gives the *kernel pair* of f. This has the structure of an internal equivalence relation in C. If f is a 'good' quotient map (in this case, a regular/effective epimorphism), then it can be recovered from this equivalence relation (as the coequalizer of its kernel pair). The situation in the 2-category of topoi is similar, but instead of an internal equivalence relation, we obtain an internal groupoid.

A geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ between topoi induces an internal groupoid in **Topos** as in the diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mathrm{pr}_{1,3}"] \arrow[r, shift right=2, "\mathrm{pr}_{1,2}"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[loop below, "\tau"] \arrow[r, shift left=2, "\mathrm{pr}_2"] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & \mathcal{F} \arrow[l, "\Delta"'] \arrow[r, "f"] & \mathcal{E}, \end{tikzcd}$$

where $\tau\colon \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is the twist map, $\Delta\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is the diagonal, and the remaining maps are the appropriate projections.

Definition VI.3. The category $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ of *descent data* for f is defined as follows.

(i) The objects of $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ are pairs (Y, θ) consisting of an object $Y \in \mathcal{F}$ and an isomorphism $\theta\colon \mathrm{pr}_1^* Y \xrightarrow{\sim} \mathrm{pr}_2^* Y$ of $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ such that

$$\Delta^*(\theta) = \mathrm{id}_X \text{ and } \mathrm{pr}_{1,3}^*(\theta) = \mathrm{pr}_{2,3}^*(\theta) \circ \mathrm{pr}_{1,2}^*(\theta).$$

This is known as a *descent datum* on Y.

(ii) A morphism $(Y, \theta) \xrightarrow{g} (Y', \theta')$ in $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ is a morphism $Y \xrightarrow{g} Y'$ of $\mathcal{F}$ such that the square

$$\begin{tikzcd} \mathrm{pr}_1^* Y \arrow[r, "\theta"] \arrow[d, "\mathrm{pr}_1^*(g)"'] & \mathrm{pr}_2^* Y \arrow[d, "\mathrm{pr}_2^*(g)"] \\ \mathrm{pr}_1^* Y' \arrow[r, "\theta'"'] & \mathrm{pr}_2^* Y' \end{tikzcd}$$

commutes.

The category $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ is a topos, and there is a *canonical functor* $c^*\colon \mathcal{E} \to \mathbf{Desc}_f(\mathcal{F}_\bullet)$ that sends an object $E \in \mathcal{E}$ to the pair consisting of f^*E and the canonical isomorphism $\mathrm{pr}_1^* f^*E \cong \mathrm{pr}_2^* f^*E$ (arising from the 2-cell of the bipullback).

In fact, Moerdijk shows in [92, §3] that the topos $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ is obtained as the colimit in the bicategory **Topos** of the diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mathrm{pr}_{1,3}"] \arrow[r, shift right=2, "\mathrm{pr}_{1,2}"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[loop below, "\tau"] \arrow[r, shift left=2, "\mathrm{pr}_2"] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & \mathcal{F} \arrow[l, "\Delta"'] \arrow[r] & \mathbf{Desc}_f(\mathcal{F}_\bullet), \end{tikzcd}$$

and the canonical functor $c^*\colon \mathcal{E} \to \mathbf{Desc}_f(\mathcal{F}_\bullet)$ is the inverse image part of the universally induced geometric morphism $\mathbf{Desc}_f(\mathcal{F}_\bullet) \to \mathcal{E}$. This is analogous to how a morphism in a 1-category factors through the coequalizer of its kernel pair.

The problem of descent involves discerning for which geometric morphisms $f\colon \mathcal{F} \to \mathcal{E}$ the canonical functor $c^*\colon \mathcal{E} \to \mathbf{Desc}_f(\mathcal{F}_\bullet)$ is an equivalence. Such geometric morphisms play the same role as regular epimorphisms did in our 1-categorical analogy.

Definition VI.4. A geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is called an *effective descent morphism* if the canonical functor $c^*\colon \mathcal{E} \to \mathbf{Desc}_f(\mathcal{F}_\bullet)$ is an equivalence.

Many examples of classes of effective descent morphisms are known, including *proper surjections* (see [63, Definition C3.2.5 & Theorem C5.1.6]). We will focus solely on *open surjections*, which were shown to be effective descent morphisms in [68, Theorem VIII.2.1], since these are the class of effective descent morphisms used in [68].

VI.2.2 Descent data with a localic domain

When the domain topos of a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is localic, say $\mathcal{F} \simeq \mathbf{Sh}(X_0)$, the category of descent data $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ is equivalent to the topos of sheaves on some localic groupoid whose locale of objects is X_0, as observed in [68, §VIII.3]. To see why this is the case, we first recall two facts about localic geometric morphisms from.

(i) Localic geometric morphisms are stable under pullback (see [59, Proposition 2.1]).

(ii) If $f\colon \mathcal{H}' \to \mathcal{H}$ is a localic geometric morphism and $\mathcal{H}$ is a localic topos, then the topos $\mathcal{H}'$ is also localic since localic geometric morphisms are closed under composition (see [59, Lemma 1.1]).

Hence, if $f\colon \mathcal{F} \to \mathcal{E}$ is a geometric morphism whose domain $\mathcal{F}$ is a localic topos, then the (bi)pullback

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd}$$

is also a localic topos, as is the wide pullback $\mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$. Therefore, as the fully faithful functor $\mathbf{Sh}\colon \mathbf{Loc} \to \mathbf{Topos}$ reflects limits, the descent diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mathrm{pr}_{1,3}" description] \arrow[r, shift right=2, "\mathrm{pr}_{1,2}"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[loop below, "\tau"] \arrow[r, shift left=2, "\mathrm{pr}_2"] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & \mathcal{F} \arrow[l, "\Delta" description] \end{tikzcd}$$

is the image under $\mathbf{Sh}$ of a localic groupoid $\mathbb{X}$

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "\mathrm{pr}_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & X_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & X_0 . \arrow[l, "e" description] \end{tikzcd} \tag{VI.i}$$

As $\mathcal{F} \simeq \mathbf{Sh}(X_0)$, an object of $\mathcal{F}$ is a local homeomorphism $q\colon Y \to X_0$, and descent datum is a morphism $\theta\colon s^*(Y) \to t^*(Y)$ in $\mathbf{Sh}(X_1) \simeq \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ such that $\mathrm{id}_{X_0} = e^*(\theta)$ and $m^*(\theta) = \pi_2^*(\theta) \circ \pi_1^*(\theta)$, i.e. the pair (Y, θ) is an object of $\mathbf{Sh}(\mathbb{X})$. Similarly, arrows in $\mathbf{Desc}_f(\mathcal{F}_\bullet)$ correspond to arrows in $\mathbf{Sh}(\mathbb{X})$. Thus, there is an equivalence $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Desc}_f(\mathcal{F}_\bullet)$ from which we obtain the following.

Theorem VI.5 (Theorem VIII.3.2 [68]). *Let $f\colon \mathbf{Sh}(X_0) \to \mathcal{E}$ be an effective descent morphism. The topos $\mathcal{E}$ is equivalent to the topos of equivariant sheaves on the localic groupoid $\mathbb{X}$ whose locale of objects is X_0, and whose source and target maps $s, t\colon X_1 \rightrightarrows X_0$ make the square*

$$\begin{tikzcd} \mathbf{Sh}(X_1) \arrow[r, "\mathbf{Sh}(s)"] \arrow[d, "\mathbf{Sh}(t)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbf{Sh}(X_0) \arrow[d, "f"] \\ \mathbf{Sh}(X_0) \arrow[r, "f"'] & \mathcal{E} \end{tikzcd}$$

a (bi)pullback of topoi.

Remark VI.6. Recall from [92, Definition 7.2] that a localic groupoid is said to be *étale complete* if the square

$$\begin{tikzcd} \mathbf{Sh}(X_1) \arrow[r, "\mathbf{Sh}(s)"] \arrow[d, "\mathbf{Sh}(t)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbf{Sh}(X_0) \arrow[d, "u"] \\ \mathbf{Sh}(X_0) \arrow[r, "u"'] & \mathbf{Sh}(\mathbb{X}) \end{tikzcd}$$

is a bipullback of topoi. This expresses that for every generalised point $x\colon U \to X_0$ and every automorphism $\alpha\colon u \circ \mathbf{Sh}(x) \overset{\sim}{\Rightarrow} u \circ \mathbf{Sh}(x)$, i.e. an automorphism of the composite

$$\mathbf{Sh}(U) \xrightarrow{\mathbf{Sh}(x)} \mathbf{Sh}(X_0) \xrightarrow{u} \mathbf{Sh}(\mathbb{X}),$$

the automorphism is instantiated by a generalised point of X_1. In other words, X_1 contains as points 'all possible automorphisms' of points of X_0.

Evidently, any representing localic groupoid of a topos constructed using the method of Theorem VI.5 will be étale complete. Indeed, we also deduce from

Theorem VI.5 that every localic groupoid for which $u\colon \mathbf{Sh}(X_0) \to \mathbf{Sh}(\mathbb{X})$ is an effective descent morphism is *Morita equivalent* to its *étale completion*, as observed in [92]. We will study a topological counterpart to the theory of étale complete localic groupoids in Section VII.5.3.

Since open surjections are effective descent morphisms, this theorem applies in particular to what we call *open covers*.

Definition VI.7. An *open cover* of the topos $\mathcal{E}$ is an open surjection $\mathcal{F} \twoheadrightarrow \mathcal{E}$ whose domain topos $\mathcal{F}$ is localic.

Remark VI.8. Recall that open geometric morphisms are stable under (bi)pullback (see [58, Theorem 4.7] or [68, Proposition VII.1.3]). Hence, if $\mathbf{Sh}(X_0) \twoheadrightarrow \mathcal{E}$ is an open cover, then the projections pr_1 and pr_2 in the (bi)pullback

$$\begin{tikzcd} \mathbf{Sh}(X_1) \simeq \mathbf{Sh}(X_0) \times_{\mathcal{E}} \mathbf{Sh}(X_0) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Sh}(X_0) \arrow[d, two heads] \\ \mathbf{Sh}(X_0) \arrow[r, two heads] & \mathcal{E}, \end{tikzcd}$$

are open too. Consequently, by [79, Proposition IX.7.2], the source and target maps $s, t\colon X_1 \rightrightarrows X_0$ of the induced localic groupoid $\mathbb{X}$ displayed in (VI.i) are open locale morphisms. Thus, $\mathcal{E}$ has an *open* representing groupoid.

The same analysis holds for any other property of geometric morphisms that is stable under pullback. For example, if an effective descent morphism $\mathbf{Sh}(X_0) \to \mathcal{E}$ is proper or connected and locally connected, then the resulting representing groupoid for $\mathcal{E}$ is also proper or connected and locally connected (in the sense that the source and target maps have these properties; see [63, Theorem C3.2.21 & Theorem C3.3.15] for a demonstration of the pullback stability of these properties).

VI.2.3 Open covers via partial equivalence relations

We are halfway to showing that every topos can be represented as the topos of sheaves on an open localic groupoid. The remaining task is to prove that every topos has an open cover.

To find an open cover of a topos $\mathcal{E}$, it suffices to find a localic geometric morphism $h\colon \mathcal{E} \to \mathcal{H}$ and an open cover $f\colon \mathcal{F} \twoheadrightarrow \mathcal{H}$, since then in the (bi)pullback

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{H}} \mathcal{E} \arrow[r, "g"] \arrow[d, two heads, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, two heads, "f"] \\ \mathcal{E} \arrow[r, "h"] & \mathcal{H}, \end{tikzcd} \tag{VI.ii}$$

the map $k\colon \mathcal{F} \times_{\mathcal{H}} \mathcal{E} \twoheadrightarrow \mathcal{E}$ is an open surjective geometric morphism whose domain is moreover a localic topos, as the composite $\mathcal{F} \times_{\mathcal{H}} \mathcal{E} \xrightarrow{g} \mathcal{F} \to \mathbf{Sets}$ is a localic morphism. Hence, $k\colon \mathcal{F} \times_{\mathcal{H}} \mathcal{E} \twoheadrightarrow \mathcal{E}$ is an open cover.

Such a pair of geometric morphisms can be found given a choice of theory classified by $\mathcal{E}$. Suppose the topos $\mathcal{E}$ is classifies a theory $\mathbb{T}$ with N sorts. Recall from Section III.4 that there is a localic geometric morphism $C_{\pi_{F\mathbb{T}}}\colon \mathcal{E} \to \mathcal{E}_{N\cdot\mathbb{O}}$ which sends a $\mathbb{T}$-model to the N underlying objects interpreting the sorts. This will play the role of h in the square (VI.ii).

Remark VI.9. In fact, we can always choose N to be 1, since every geometric theory $\mathbb{T}$ is Morita equivalent to a single-sorted theory. This appears in [68] as Proposition VII.3.1, but an entirely syntactic proof is given in [63, Lemma D1.4.13]. In summary, the idea is to combine all the sorts of the theory into one, and introduce new unary relation symbols, R^X for each sort X, and axioms such that $R^X(x)$ expresses the statement "x belongs to the sort X".

We must now describe an open cover of $\mathcal{E}_{N\cdot\mathbb{O}}$ to play the role of f in (VI.ii). As prefigured in Example V.19, there is a sense in which 'every set is a subquotient of $\mathbb{N}$' and so we are motivated to consider partial equivalence relations on $\mathbb{N}$. We denote the classifying topos of partial equivalence relations on N copies of $\mathbb{N}$ by $\mathcal{E}_{N\cdot\mathcal{P}Q_{\mathbb{N}}}$. This is the propositional theory whose basic propositions are $[n \sim^i m]$ for each $n, m \in \mathbb{N}$, and $i \in N$ (meaning that n, m are identified in the ith partial equivalence relation on $\mathbb{N}$), and whose axioms are

$$\begin{aligned} [n \sim^i m] &\vdash [m \sim^i n] && \text{(symmetry)} \\ [n \sim^i l] \wedge [l \sim^i m] &\vdash [n \sim^i m] && \text{(transitivity)} \end{aligned}$$

for each $n, m, l \in \mathbb{N}$ and $i \in N$. Being a propositional theory, the classifying topos $\mathcal{E}_{N\cdot\mathcal{P}Q_{\mathbb{N}}}$ is localic.

There is a geometric morphism $Q\colon \mathcal{E}_{N\cdot\mathcal{P}Q_{\mathbb{N}}} \to \mathcal{E}_{N\cdot\mathbb{O}}$ that sends the N generic partial equivalence relations on $\mathbb{N}$ to their corresponding subquotient objects. This geometric morphism possesses many desirable properties: it is open and surjective, but also connected and locally connected (see [63, Theorem C5.2.7]). Hence, we indeed have an open cover of $\mathcal{E}_{N\cdot\mathbb{O}}$.

We now obtain an open cover $\overline{P}_N[\mathcal{E}] \twoheadrightarrow \mathcal{E}$ by taking the (bi)pullback

$$\begin{tikzcd} \overline{P}_N[\mathcal{E}] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_{N\cdot\mathcal{P}Q_{\mathbb{N}}} \arrow[d, two heads, "Q"] \\ \mathcal{E} \arrow[r, "C_{\pi_F\mathbb{T}}"] & \mathcal{E}_{N\cdot\mathbb{O}} \end{tikzcd}$$

Note that $\overline{P}_N[\mathcal{E}]$ is determined not only by $\mathcal{E}$, but also by the localic geometric morphism $\mathcal{E} \to \mathcal{E}_{N\cdot\mathbb{O}}$, and hence by a choice of N-sorted geometric theory $\mathbb{T}$ classified by $\mathcal{E}$ (by Proposition III.42). In Lemma VI.19, we describe a propositional theory classified by the topos $\overline{P}_N[\mathcal{E}]$.

Finally, as every topos classifies some geometric theory, by applying Theorem VI.5 we arrive at the landmark result of Joyal and Tierney.

Theorem VI.10 (Theorem VIII.3.2 [68])**.** *Every Grothendieck topos can be represented as the topos of equivariant sheaves for a localic groupoid.*

Remarks VI.11. (i) Since the geometric morphism Q above is open (and even connected and locally connected), the representing localic groupoid is also open (indeed, connected and locally connected; see Remark VI.8).

(ii) A topos can have many non-equivalent open covers and therefore many non-isomorphic representing localic groupoids. Nonetheless, these are all equivalent in a suitable sense provided by [92, §7] (see also Section VIII.1).

The open cover $\overline{P}_N[\mathcal{E}] \twoheadrightarrow \mathcal{E}$ we consider is slightly different to the one built by Joyal and Tierney in [68, Theorem VII.3.1]. They instead use the open cover $\mathcal{E}_{\mathcal{TQ}_{\mathbb{N}}} \twoheadrightarrow \mathcal{E}_{\mathbb{O}_{>0}}$ from classifying topos of *total* equivalence relations on $\mathbb{N}$ to the classifying topos of *inhabited* objects. The reader is directed to [63, Remark C5.2.8(c)] for more details. Other examples of open covers include the *Diaconescu cover*, constructed in [33] (see also [63, Theorem C5.2.1] and [79, Theorem IX.9.1]). See also Remark VI.23.

VI.3 The syntactic groupoid

Let $\mathbb{T}$ be a geometric theory. We give an explicit description of the representing localic groupoid for the classifying topos $\mathcal{E}_{\mathbb{T}}$ via Theorem VI.10, which we call the *syntactic groupoid* because of its obvious syntactic nature.

VI.3.1 Description of the syntactic groupoid

The syntactic groupoid $\mathbb{G}^{\mathbb{T}}$ is motivated by a desire to re-express the first-order theory $\mathbb{T}$ in terms of simpler propositional theories. The models of this new propositional theory should somehow represent the models of the original theory $\mathbb{T}$, including the objects being used to represent each sort. The question then is how to encode the sorts of $\mathbb{T}$ using only propositional logic.

Sorts as partial equivalence relations. If we were to focus on a single set-based model M, then we could include propositional variables in our language that express that $\vec{m} \in R^M$ for each relation R of the theory and each appropriate tuple $\vec{m}$ of elements from M. More generally, we could imagine fixing some a suitably large set $\mathfrak{K}$ and cutting out the carriers for each model as subsets or subquotients of $\mathfrak{K}$. The issue is that in general a geometric theory has unboundedly large models.

However, recall from Example V.19 that, although the topological space of partial surjections from $\mathbb{N}$ to any set X might be trivial, the localic version is not. Hence, there is a sense in which 'every set is a subquotient of $\mathbb{N}$'. This motivates replacing the sorts in the theory $\mathbb{T}$ by partial equivalence relations on $\mathbb{N}$, which describe these subquotients. Recall that a partial equivalence relation is a symmetric transitive relation and can be thought of as describing an equivalence relation on the subset of elements which are related to themselves.

Definition VI.12. Let $\mathbb{T}$ be a theory over a signature Σ without function symbols (if $\mathbb{T}$ involves function symbols, these can be removed by [63, Lemma D1.4.9]). We define the propositional geometric theory $P[\mathbb{T}]$ over the signature $P[\Sigma]$ as follows.

(i) For each sort X of Σ, we add a copy of the theory of partial equivalence relations on $\mathbb{N}$. Explicitly, we add, for each $n, m \in \mathbb{N}$, a basic proposition $[n \sim^X m]$ to $P[\Sigma]$

and, for each $n, m, l \in \mathbb{N}$, and the sequents

$$[n \sim^X m] \vdash [m \sim^X n], \qquad \text{(symmetry)}$$
$$[n \sim^X m] \wedge [m \sim^X l] \vdash [n \sim^X l] \qquad \text{(transitivity)}$$

to the axioms of $P[\mathbb{T}]$.

(ii) For each relation symbol $R \subseteq X^1 \times \cdots \times X^k$ of Σ, and for each pair of tuples $n_1, \ldots, n_k \in \mathbb{N}$ and $m_1, \ldots, m_k \in \mathbb{N}$, we add a proposition $[(n_1, \ldots, n_k) \in R]$ to $P[\Sigma]$ and the sequents

$$[(n_1, \ldots, n_k) \in R] \wedge [n_1 \sim^{X^1} m_1] \wedge \cdots \wedge [n_k \sim^{X^k} m_k] \vdash [(m_1, \ldots, m_k) \in R],$$

and

$$[(n_1, \ldots, n_k) \in R] \vdash [n_1 \sim^{X^1} n_1] \wedge \cdots \wedge [n_k \sim^{X^k} n_k]$$

as axioms to $P[\mathbb{T}]$.

(iii) For each axiom $\varphi \vdash_{x_1:X^1,\ldots,x_k:X^k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\ldots,n_k} \vdash \psi_{n_1,\ldots,n_k}$$

for each $n_1, \ldots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\ldots,n_k}$ and $\psi_{n_1,\ldots,n_k}$ are obtained from φ and ψ by

a) replacing each free variable x_i by a (fixed) natural number n_i,
b) each quantifier $\exists x : X\, \chi(x, \ldots)$ by a join $\bigvee_{n_x \in \mathbb{N}} \chi(n_x, \ldots)$,
c) each subformula of the form $R(y_1, \ldots, y_l)$ with $[(y_1, \ldots, y_l) \in R]$,
d) and each subformula of the form $x =_X y$ with $[x \sim^X y]$.

We denote the classifying locale of the propositional theory $P[\mathbb{T}]$ by $G_0^{\mathbb{T}}$.

Here we have simply translated the relations on the sorts to relations on $\mathbb{N}$ that respect the partial equivalence relation. We have written the axioms of $\mathbb{T}$ in terms of these with existential quantification over sorts being expressed using joins over the natural numbers. Evidently, if $\mathbb{T}$ is a propositional theory (i.e. there are no sorts), then $\mathbb{T}$ and $P[\mathbb{T}]$ are the same theory.

Remark VI.13. Note that the generators $[n \sim^X m]$ can also be thought of as a as special case of the proposition $[(n, m) \in R]$, where R is given by the equality relation on X.

Encoding isomorphic copies. The points of the locale $G_0^{\mathbb{T}}$ are given by representations of models of $\mathbb{T}$ as subquotients of $\mathbb{N}$. Different subquotients of $\mathbb{N}$ might correspond to isomorphic models, so these must be identified in the locale of isomorphisms of the syntactic groupoid.

We can define a geometric theory $\mathbb{T}_{\cong}$ whose models are isomorphisms between models of $\mathbb{T}$, and then transform it into a propositional theory as we did for $\mathbb{T}$ in Definition VI.12. The theory $\mathbb{T}_{\cong}$ is precisely the theory classified by the iso-comma object described in Remark VI.2.

Definition VI.14. Given a geometric theory $\mathbb{T}$ over a signature Σ (where we again assume that there are no function symbols), we construct $\mathbb{T}_{\cong}$ as the following geometric theory over the signature $\Sigma_{\cong}$.

(i) For each sort X or relation symbol R of Σ, we add a pair of copies X_1, X_2 or R_1, R_2 to $\Sigma_{\cong}$ (where R_i is defined on the i-subscripted sorts).

(ii) For each axiom of $\varphi \vdash_{\vec{x}} \psi$ of $\mathbb{T}$, we add a pair of sequents $\varphi_1 \vdash_{\vec{x}_1} \psi_1$ and $\varphi_2 \vdash_{\vec{x}_2} \psi_2$ to $\mathbb{T}_{\cong}$, where the formulae φ_i, ψ_i are obtained by replacing each variable of sort X with a variable of sort X_i, and each relation symbol R with the relation symbol R_i.

(iii) For each sort X in Σ, we add a relation symbol $\alpha^X \subseteq X_1 \times X_2$ to $\Sigma_{\cong}$ together with the bidirectional sequent

$$(x, y) \in \alpha^X \wedge (x', y') \in \alpha^X \wedge x =_{X_1} x' \dashv\vdash_{x:X_1, y:X_2} (x, y) \in \alpha^X \wedge (x', y') \in \alpha^X \wedge y =_{X_2} y'$$

along with the sequents

$$\vdash_{y:X_2} \exists x : X_1 \ (x, y) \in \alpha^X,$$
$$\vdash_{x:X_1} \exists y : X_2 \ (x, y) \in \alpha^X,$$

as axioms to $\mathbb{T}_{\cong}$, making α into the graph of a bijection[1].

(iv) For each relation symbol R of Σ, we add to $\mathbb{T}_{\cong}$ the bidirectional axiom

$$\bigwedge_{i=1}^{k} (x_i, y_i) \in \alpha^{X^i} \wedge (x_1, \ldots, x_k) \in R_1 \dashv\vdash_{x_1, \ldots, x_k, y_1, \ldots, y_k} \bigwedge_{i=1}^{k} (x_i, y_i) \in \alpha^{X^i} \wedge (y_1, \ldots, y_k) \in R_2$$

expressing that the bijection encoded by α is an isomorphism of Σ-structures.

We define the locale $G_1^{\mathbb{T}}$ to be the classifying locale of a propositional geometric theory $P[\mathbb{T}_{\cong}]$ defined as in Definition VI.12.

Remark VI.15. Let $\mathbb{T}$ be a propositional geometric theory over a signature Σ. Condition (iv) from Definition VI.14 entails that the copies R_1 and R_2 of each basic proposition R in Σ are equivalent. Thus, the theories $\mathbb{T}$, $P[\mathbb{T}]$ and $P[\mathbb{T}_{\cong}]$ are all equivalent.

Structural morphisms of the syntactic groupoid. The localic groupoid $\mathbb{G}^{\mathbb{T}}$ has $G_0^{\mathbb{T}}$ as its locale of objects and $G_1^{\mathbb{T}}$ as is locale of morphisms. We now describe the structural morphisms of the groupoid $\mathbb{G}^{\mathbb{T}}$. Recall that it is possible to define a frame homomorphism, and hence a locale morphism, by specifying its action on generators. In the case of $G_0^{\mathbb{T}}$ and $G_1^{\mathbb{T}}$, this amounts to defining the action on the basic propositions of the propositional theories $P[\mathbb{T}]$ and $P[\mathbb{T}_{\cong}]$.

Definition VI.16. Let $\mathbb{G}^{\mathbb{T}}$ denote the localic groupoid

$$\begin{tikzcd} G_1^{\mathbb{T}} \times_{G_0^{\mathbb{T}}} G_1^{\mathbb{T}} \arrow[r, "m"] & G_1^{\mathbb{T}} \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & G_0^{\mathbb{T}} \arrow[l, "e" description] \end{tikzcd}$$

whose morphisms are defined as follows.

[1]For clarity we will often write α suggestively as though it were a function.

(i) The source map $s\colon G_1^{\mathbb{T}} \to G_0^{\mathbb{T}}$ is specified by frame homomorphism that acts on generators by

$$[n \sim^X m] \mapsto [n \sim^{X_1} m],$$
$$[(n_1, \ldots, n_k) \in R] \mapsto [(n_1, \ldots, n_k) \in R_1].$$

(ii) Similarly, the target map $t\colon G_1^{\mathbb{T}} \to G_0^{\mathbb{T}}$ is specified by the action on generators

$$[n \sim^X m] \mapsto [n \sim^{X_2} m],$$
$$[(n_1, \ldots, n_k) \in R] \mapsto [(n_1, \ldots, n_k) \in R_2].$$

(iii) The identity map $e\colon G_0^{\mathbb{T}} \to G_1^{\mathbb{T}}$ is specified by

$$[(n_1, \ldots, n_k) \in R_1] \mapsto [(n_1, \ldots, n_k) \in R],$$
$$[(n_1, \ldots, n_k) \in R_2] \mapsto [(n_1, \ldots, n_k) \in R],$$
$$[\alpha^X(n) = m] \mapsto [n \sim^X m].$$

(iv) The inversion map $i\colon G_1^{\mathbb{T}} \to G_1^{\mathbb{T}}$ swaps the two copies of the sorts in the sense that

$$[(n_1, \ldots, n_k) \in R_1] \mapsto [(n_1, \ldots, n_k) \in R_2],$$
$$[(n_1, \ldots, n_k) \in R_2] \mapsto [(n_1, \ldots, n_k) \in R_1],$$
$$[\alpha^X(n) = m] \mapsto [\alpha^X(m) = n].$$

(v) The composition map $m\colon G_1^{\mathbb{T}} \times_{G_0^{\mathbb{T}}} G_1^{\mathbb{T}} \to G_1^{\mathbb{T}}$ is given as follows.

a) The domain of the composition map can be alternatively described as the classifying locale of the propositional geometric theory $P[\mathbb{T}_{\cong,\cong}]$, where $\mathbb{T}_{\cong,\cong}$ is the geometric theory whose models are a triple of $\mathbb{T}$-models and a pair of isomorphisms between these (cf. Remark VI.2).

The theory $\mathbb{T}_{\cong,\cong}$ is constructed much like $\mathbb{T}_{\cong}$, but there are three copies of the theory $\mathbb{T}$ instead of two and there are two relation symbols $\beta^X \subseteq X_1 \times X_2$ and $\gamma^X \subseteq X_2 \times X_3$ for each sort X, encoding two $\mathbb{T}$-model isomorphisms, instead of one relation symbol α^X.

b) The map m itself is specified by the action

$$[(n_1, \ldots, n_k) \in R_1] \mapsto [(n_1, \ldots, n_k) \in R_1],$$
$$[(n_1, \ldots, n_k) \in R_2] \mapsto [(n_1, \ldots, n_k) \in R_3],$$
$$[\alpha^X(n) = p] \mapsto \bigvee_{m \in \mathbb{N}} [\beta^X(n) = m] \wedge [\gamma^X(m) = p],$$

i.e. the map m sends the pair of relations (β^X, γ^X) to their relational composite.

Remark VI.17. The set $\mathbb{N}$ is actually only the simplest possible choice of base set for the above construction. All the properties we prove of the localic groupoid $\mathbb{G}^{\mathbb{T}}$ (other than those discussed in Section VI.3.3) will still hold if $\mathbb{N}$ is replaced with any infinite set.

VI.3.2 The syntactic groupoid is representing

We now prove that the localic groupoid $\mathbb{G}^{\mathbb{T}}$ described in Definition VI.16 is the representing localic groupoid for the topos $\mathcal{E}_{\mathbb{T}}$ yielded by the Joyal-Tierney method exposited in Section VI.2, that is we will prove that:

Theorem VI.18. *For each geometric theory $\mathbb{T}$, there is an equivalence of topoi*

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}\left(\mathbb{G}^{\mathbb{T}}\right).$$

We require one lemma before embarking on the proof of the theorem.

Lemma VI.19. *For each geometric theory $\mathbb{T}$ with N sorts, the commutative square*

$$\begin{tikzcd}
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] \arrow[d, two heads] & \mathcal{E}_{N \cdot \mathcal{P}Q_{\mathbb{N}}} \arrow[d, two heads, "Q"] \\
\mathcal{E}_{\mathbb{T}} \arrow[r, "C_{\pi_F \mathbb{T}}"'] & \mathcal{E}_{N \cdot \mathbb{O}}
\end{tikzcd}$$

is a (bi)pullback, i.e. $\overline{P}_N[\mathcal{E}] \simeq \mathcal{E}_{P[\mathbb{T}]}$.

Proof. For simplicity, we will assume the theory $\mathbb{T}$ has a single sort, but this is easily generalised. Recall that $Q\colon \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \to \mathcal{E}_{\mathbb{O}}$ is the geometric morphism that acts on models by sending a partial equivalence relation $\sim$ on $\mathbb{N}$ to the corresponding subquotient $\mathbb{N}/{\sim}$. As described in Examples VI.1, it is easy to compute a theory $\mathbb{T}'$ that the bipullback topos $\mathcal{E}_{\mathbb{T}} \times_{\mathcal{E}_{\mathbb{O}}} \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}}$ classifies using the methods of [123, §4.5]. The theory $\mathbb{T}'$ can be taken to be the theory of pairs of a model M of $\mathbb{T}$, a model $\sim$ of $\mathcal{P}Q_{\mathbb{N}}$ and an isomorphism $M \cong \mathbb{N}/{\sim}$.

It is now elementary to massage $\mathbb{T}'$ into a more convenient, equivalent form by transporting the $\mathbb{T}$-model structure on M along the bijection $M \cong \mathbb{N}/\sim$ to give relations defined in $\mathbb{N}/{\sim}$. Then, since the object M and its definable subobjects are completely specified by relations on $\mathbb{N}/{\sim}$ and the isomorphism $M \cong \mathbb{N}/{\sim}$, the isomorphism can safely be removed from the theory. The resulting theory is essentially propositional. We can make it manifestly propositional by replacing a relation R on $\mathbb{N}/{\sim}$ with its preimages under the quotient $\mathbb{N} \twoheadrightarrow \mathbb{N}/{\sim}$ to give a subset $U_R \subseteq \mathbb{N}^k$, which can then be described using the basic generators

$$[(n_1, \dots, n_k) \in U_R]$$

for each $(n_1, \dots, n_k) \in \mathbb{N}^k$. Thus, we have arrived at the theory $P[\mathbb{T}]$ described in Definition VI.12. This theory now has no sorts and so it is manifestly propositional. $\square$

Proof of Theorem VI.18. Again we assume that $\mathbb{T}$ has one sort for simplicity. By Lemma VI.19, the open cover $\overline{P}[\mathcal{E}_{\mathbb{T}}] \twoheadrightarrow \mathcal{E}_{\mathbb{T}}$ used to construct the representing groupoid in Section VI.2.3 is the projection from the bipullback

$$\mathbf{Sh}(G_0^{\mathbb{T}}) \simeq \mathcal{E}_{P[\mathbb{T}]} \simeq \mathcal{E}_{\mathbb{T}} \times_{\mathbb{T}_{\mathbb{O}}} \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{E}_{\mathbb{T}}.$$

By applying Theorem VI.5, we know that $\mathcal{E}_{\mathbb{T}}$ is represented by the localic groupoid whose locale of objects is $G_0^{\mathbb{T}}$, the classifying locale of $P[\mathbb{T}]$, and whose source and target maps $s, t\colon Y \rightrightarrows G_0^{\mathbb{T}}$ are the locale morphisms for which the square

$$\begin{tikzcd}
\mathbf{Sh}(Y) \arrow[r, "\mathbf{Sh}(s)"] \arrow[d, "\mathbf{Sh}(t)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}}
\end{tikzcd}$$

is a bipullback of topoi. We must show that $\mathcal{E}_{P[\mathbb{T}_{\cong}]} \simeq \mathbf{Sh}(G_1^{\mathbb{T}})$ is this bipullback.

Let $2 \cdot \mathbb{O}$ denote the theory of pairs of objects, which by Examples VI.1 is classified by the product $2 \cdot \mathbb{O}$ is classified by the product $\mathcal{E}_{2\cdot\mathbb{O}} \cong \mathcal{E}_{\mathbb{O}} \times \mathcal{E}_{\mathbb{O}}$. Similarly, the product $\mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \times \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}}$ classifies the theory $2 \cdot \mathcal{P}Q_{\mathbb{N}}$ of pairs of partial equivalence relations on $\mathbb{N}$. Recall also from Examples VI.1 and Remark VI.2 that the theory $\mathbb{T}_{\cong}$ of isomorphisms of $\mathbb{T}$-models is classified by the bipullback

$$\begin{tikzcd}
\mathcal{E}_{\mathbb{T}_{\cong}} \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] \\
\mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] & \mathcal{E}_{\mathbb{T}}.
\end{tikzcd}$$

By the universal property of $\mathcal{E}_{P[\mathbb{T}]}$, there are universally induced geometric morphisms $s, t\colon \mathcal{E}_{P[\mathbb{T}_{\cong}]} \rightrightarrows \mathcal{E}_{P[\mathbb{T}]}$ such that all the squares in the diagram

$$\begin{tikzcd}
 & & \mathcal{E}_{P[\mathbb{T}]} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \arrow[dd] \\
 & \mathcal{E}_{P[\mathbb{T}_{\cong}]} \arrow[ur, dashed, "s"] \arrow[rr] \arrow[dl, dashed, "t"'] \arrow[ddd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{E}_{2\cdot\mathcal{P}Q_{\mathbb{N}}} \arrow[ur] \arrow[dl] \arrow[ddd] & \\
 & & \mathcal{E}_{\mathbb{T}} \arrow[rr] & & \mathcal{E}_{\mathbb{O}} \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \arrow[dd] & & \\
 & \mathcal{E}_{\mathbb{T}_{\cong}} \arrow[rr] \arrow[uur, "r"] \arrow[dl, "u"'] & & \mathcal{E}_{2\cdot\mathbb{O}} \arrow[uur] \arrow[dl] & \\
\mathcal{E}_{\mathbb{T}} \arrow[rr] & & \mathcal{E}_{\mathbb{O}} & &
\end{tikzcd}$$

commute up to canonical isomorphisms. Being induced by the maps $r, u\colon \mathcal{E}_{\mathbb{T}_{\cong}} \rightrightarrows \mathcal{E}_{\mathbb{T}}$, which send a model a $\mathbb{T}_{\cong}$-model $M \cong N$ to, respectively, M and N, we recognise that the locale morphisms $s, t\colon G_1^{\mathbb{T}} \rightrightarrows G_0^{\mathbb{T}}$ corresponding to the geometric morphisms $s, t\colon \mathcal{E}_{P[\mathbb{T}_{\cong}]} \rightrightarrows \mathcal{E}_{P[\mathbb{T}]}$ are exactly the ones described in Definition VI.16. Note that we are abusing notation and not differentiating between a locale morphism and its corresponding geometric morphism between localic topoi.

We now demonstrate that the square

$$\begin{tikzcd}
\mathcal{E}_{P[\mathbb{T}_\cong]} \arrow[r, "s"] \arrow[d, "t"'] & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}}
\end{tikzcd}$$

is a bipullback of topoi. Firstly, we note that the square commutes up to isomorphism since it can be rewritten as

$$\begin{tikzcd}
\mathcal{E}_{P[\mathbb{T}_\cong]} \arrow[rr, "s"] \arrow[dr] \arrow[dd, "t"'] & & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
& \mathcal{E}_{\mathbb{T}_\cong} \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_\mathbb{T}}"] \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_\mathbb{T}}"'] & \mathcal{E}_{\mathbb{T}}.
\end{tikzcd}$$

For any other (bi)cone

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}}
\end{tikzcd}$$

of the cospan, we will demonstrate that there is a diagram of topoi and geometric morphisms

$$\begin{tikzcd}
& & & \mathcal{E}_{P[\mathbb{T}]} \arrow[rr] \arrow[ddd] & & \mathcal{E}_{\mathcal{P}Q_\mathbb{N}} \arrow[ddd] \\
\mathcal{F} \arrow[urrr, bend left, "f"] \arrow[drrrr, dashed, bend left] \arrow[drr, dotted] \arrow[dddr, bend right, "g"'] \arrow[ddddrr, dashed] & & & & & \\
& & \mathcal{E}_{P[\mathbb{T}_\cong]} \arrow[uur] \arrow[rr] \arrow[ddl] \arrow[ddd] & & \mathcal{E}_{2\cdot\mathcal{P}Q_\mathbb{N}} \arrow[uur] \arrow[ddl] \arrow[ddd] & \\
& & & \mathcal{E}_{\mathbb{T}} \arrow[rr, dashed] & & \mathcal{E}_{\mathbb{O}} \\
& \mathcal{E}_{P[\mathbb{T}]} \arrow[rr] \arrow[dd] & & \mathcal{E}_{\mathcal{P}Q_\mathbb{N}} \arrow[dd] & & \\
& & \mathcal{E}_{\mathbb{T}_\cong} \arrow[uur] \arrow[rr] \arrow[dl] & & \mathcal{E}_{2\cdot\mathbb{O}} \arrow[uur] \arrow[dl] & \\
& \mathcal{E}_{\mathbb{T}} \arrow[rr] & & \mathcal{E}_{\mathbb{O}} & &
\end{tikzcd}$$

where every square and triangle commutes up to canonical isomorphism.

(i) The geometric morphism $\mathcal{F} \dashrightarrow \mathcal{E}_{\mathbb{T}_{\cong}}$ is induced by the universal property of $\mathcal{E}_{\mathbb{T}_{\cong}}$ as in the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, dashed] \arrow[dd, "g"'] & \arrow[d, phantom, "\cong"] & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
\arrow[r, phantom, "\cong"] & \mathcal{E}_{\mathbb{T}_{\cong}} \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner\ \cong"] & \mathcal{E}_{\mathbb{T}} \arrow[d, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"] \\
\mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}} \arrow[r, "\mathrm{id}_{\mathcal{E}_{\mathbb{T}}}"'] & \mathcal{E}_{\mathbb{T}}.
\end{tikzcd}$$

(ii) The geometric morphism $\mathcal{F} \dashrightarrow \mathcal{E}_{2\cdot\mathcal{P}Q_{\mathbb{N}}}$ is induced by the universal property of $\mathcal{E}_{2\cdot\mathcal{P}Q_{\mathbb{N}}}$ as in the diagram

$$\begin{tikzcd}
\mathcal{E}_{P[\mathbb{T}]} \arrow[dd] & \mathcal{F} \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, dashed] & \mathcal{E}_{P[\mathbb{T}]} \arrow[dd] \\
\arrow[r, phantom, "\cong"] & \mathcal{E}_{2\cdot\mathcal{P}Q_{\mathbb{N}}} \arrow[d, phantom, "\wr"] \arrow[r, phantom, "\cong"] & {} \\
\mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} & \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \times \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}} \arrow[l] \arrow[r] & \mathcal{E}_{\mathcal{P}Q_{\mathbb{N}}}
\end{tikzcd}$$

(iii) Finally, the geometric morphism $\mathcal{F} \dashrightarrow \mathcal{E}_{P[\mathbb{T}_{\cong}]}$ is induced by the universal property of $\mathcal{E}_{P[\mathbb{T}_{\cong}]}$ as in the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, dotted] \arrow[drr, dashed, bend left] \arrow[ddr, dashed, bend right] & \arrow[d, phantom, "\cong"] & \\
\arrow[r, phantom, "\cong"] & \mathcal{E}_{P[\mathbb{T}_{\cong}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \cong"] & \mathcal{E}_{2\cdot\mathcal{P}Q_{\mathbb{N}}} \arrow[d] \\
& \mathcal{E}_{\mathbb{T}_{\cong}} \arrow[r] & \mathcal{E}_{2\cdot\mathbb{O}}.
\end{tikzcd}$$

Thus, the (bi)cone factorises canonically as

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, dotted] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & \arrow[d, phantom, "\cong"] & \\
\arrow[r, phantom, "\cong"] & \mathcal{E}_{P[\mathbb{T}_{\cong}]} \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{P[\mathbb{T}]} \arrow[d] \\
& \mathcal{E}_{P[\mathbb{T}]} \arrow[r] & \mathcal{E}_{\mathbb{T}}.
\end{tikzcd}$$

We have elided the details that $\mathcal{E}_{P[\mathbb{T}_{\cong}]}$ also satisfies the necessary universal property on 2-cells to be the bipullback, but this can be demonstrated in a similar fashion since the canonical morphism $\mathcal{F} \dashrightarrow \mathcal{E}_{P[\mathbb{T}_{\cong}]}$ was universally induced by a series of bilimits.

Finally, by demonstrating in an analogous manner that $\mathcal{E}_{P[\mathbb{T}_{\cong,\cong}]}$ is equivalent to the wide bipullback $\mathcal{E}_{P[\mathbb{T}]} \times_{\mathcal{E}_{\mathbb{T}}} \mathcal{E}_{P[\mathbb{T}]} \times_{\mathcal{E}_{\mathbb{T}}} \mathcal{E}_{P[\mathbb{T}]}$, we recognise that the composition map of our groupoid is described as in Definition VI.16, thus completing the proof that the localic groupoid $\mathbb{G}^{\mathbb{T}}$ represents $\mathcal{E}_{\mathbb{T}}$. □

Example VI.20. As remarked in Remark VI.15, when $\mathbb{T}$ is a propositional theory, the theories $\mathbb{T}$, $P[\mathbb{T}]$ and $P[\mathbb{T}_{\cong}]$ are all equivalent, and therefore have isomorphic classifying locales. Hence, the syntactic groupoid $\mathbb{G}^{\mathbb{T}}$ as described in Definition VI.16 is an example of a localic groupoid of the form Examples V.3(i) and so there is an equivalence $\mathbf{Sh}(\mathbb{G}^{\mathbb{T}}) \simeq \mathbf{Sh}(G_0^{\mathbb{T}})$, i.e. the classifying topos for $\mathbb{T}$ is equivalent to the topos of sheaves on the classifying locale of $\mathbb{T}$, as we would expect.

VI.3.3 Countable theories and Forssell groupoids

Recall from Section V.2 that if the locale of objects $G_0^{\mathbb{T}}$, the locale of morphisms $G_1^{\mathbb{T}}$ and the locale of composable morphisms $G_1^{\mathbb{T}} \times_{G_0^{\mathbb{T}}} G_1^{\mathbb{T}}$ are spatial, then there is an equivalence of topoi

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{G}^{\mathbb{T}}) \simeq \mathbf{Sh}(\mathrm{Pt}(\mathbb{G}^{\mathbb{T}})),$$

where $\mathrm{Pt}(\mathbb{G}^{\mathbb{T}})$ is the corresponding topological groupoid. We can ensure these spatiality conditions under certain countability restrictions on the theory.

Definition VI.21. A geometric theory is said to be *countable* if it has a countable number of sorts, symbols and axioms.

Proposition VI.22. *For a countable geometric theory $\mathbb{T}$, the localic groupoid $\mathbb{G}^{\mathbb{T}}$ is spatial and thus arises from a topological groupoid.*

Proof. Note that if a theory $\mathbb{T}$ is countable, then the locale of objects $G_0^{\mathbb{T}}$ and the locale of morphisms $G_1^{\mathbb{T}}$ of the representing localic groupoid are countably presented. A countably presented locale is spatial (assuming the excluded middle, see [49, Corollary 3.14]). Moreover, since countably presented locales are closed under finite limits, the domain of the composition map is also spatial, as required. □

When $\mathbb{T}$ is a countable geometric theory, the topological groupoid $\mathrm{Pt}(\mathbb{G}^{\mathbb{T}})$ thus obtained is the same representing topological groupoid as constructed by Forssell in [37]. The representation of classifying topoi by topological groupoids is studied in Chapter VII, and *Forssell groupoids*, in particular, are discussed in Section VII.5.4. While we further elucidate the connection between $\mathrm{Pt}(\mathbb{G}^{\mathbb{T}})$ and Forssell groupoids, we adopt the as-yet unintroduced terminology and notation from Chapter VII.

The Forssell groupoid $\mathcal{FG}(\mathbb{N})$ (see [37, §3] or Definition VII.53) is the topological groupoid

$$\begin{tikzcd} \mathcal{FG}(\mathbb{N})_1 \times_{\mathcal{FG}(\mathbb{N})_0} \mathcal{FG}(\mathbb{N})_1 \arrow[r, "m"] & \mathcal{FG}(\mathbb{N})_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & \mathcal{FG}(\mathbb{N})_0 \arrow[l, "e"'] \end{tikzcd}$$

constructed as follows.

(i) The *space of* $\mathbb{N}$*-indexed models* $\mathcal{FG}(\mathbb{N})_0$ is the set of all $\mathbb{N}$-indexed models of $\mathbb{T}$, i.e. those set-based models of $\mathbb{T}$ whose underlying sets of each sort are subquotients of $\mathbb{N}$. For a tuple $\vec{n} \in \mathbb{N}$, we denote by $[\vec{n}]$ its equivalence class in M. Since $\mathbb{T}$ is a countable theory, by [87, Theorem 6.2.4] (and the downward Löwenheim-Skolem theorem if necessary), the set $\mathcal{FG}(\mathbb{N})_0$ is a *conservative* set of models.

We endow the set $\mathcal{FG}(\mathbb{N})_0$ with the *logical topology for objects* (see [37, Definition 3.1] or Definition VII.14), the topology generated by subsets of form

$$[\![\vec{n} \in R]\!]_{\mathcal{FG}(\mathbb{N})} = \left\{ M \in \mathcal{FG}(\mathbb{N})_0 \,\middle|\, [\vec{n}] \in R^M \right\},$$

where R is a relation of $\mathbb{T}$ (including equality), R^M is its interpretation in a model M, and $\vec{n}$ is a tuple of natural numbers.

We immediately recognise the frame of opens $\mathcal{O}(\mathcal{FG}(\mathbb{N})_0)$ as the frame $G_0^{\mathbb{T}}$ from Definition VI.12. Explicitly, we identify the basic open

$$[\![\vec{n} \in R]\!]_{\mathcal{FG}(\mathbb{N})} \subseteq \mathcal{FG}(\mathbb{N})_0$$

with the generator $[\vec{n} \in R]$ of $G_0^{\mathbb{T}}$.

(ii) The space of arrows $\mathcal{FG}(\mathbb{N})_1$ is the set of all isomorphisms between models in $\mathcal{FG}(\mathbb{N})_0$ endowed with the *logical topology for arrows* (see [37, Definition 3.1] or Definition VII.19), the topology generated by sets of the form

$$\left[\!\!\left[\begin{matrix} \vec{n} \in R \\ \vec{m} \mapsto \vec{m}' \\ \vec{n}' \in R' \end{matrix} \right]\!\!\right]_{\mathcal{FG}(\mathbb{N})} = \left\{ M \xrightarrow{\alpha} M' \in \mathcal{FG}(\mathbb{N})_1 \,\middle|\, \begin{matrix} [\vec{n}] \in R^M, \\ [\vec{m}] \in M,\ [\vec{m}'] \in M', \\ \alpha([\vec{m}]) = [\vec{m}'], \\ [\vec{n}'] \in R'^{M'}. \end{matrix} \right\}$$

Once again, we identify $\mathcal{O}(\mathcal{FG}(\mathbb{N})_1)$ with $G_1^{\mathbb{T}}$ by identifying the basic open

$$\left[\!\!\left[\begin{matrix} \vec{n} \in R \\ \vec{m} \mapsto \vec{m}' \\ \vec{n}' \in R' \end{matrix} \right]\!\!\right]_{\mathcal{FG}(\mathbb{N})} \subseteq \mathcal{FG}(\mathbb{N})$$

with $[\vec{n} \in R_1] \wedge [\vec{n}' \in R_2'] \wedge \bigwedge_{m_i \in \vec{m}} [\alpha(m_i) = m_i'] \in \mathbb{G}_1^{\mathbb{T}}$.

(iii) The maps m, t, e, s and i are defined in the obvious way.

Thus, when the theory $\mathbb{T}$ is countable, the syntactic localic groupoid $\mathbb{G}^{\mathbb{T}}$ as constructed in Definition VI.16 coincides with the topological groupoid $\mathcal{FG}(\mathbb{N})$ of $\mathbb{N}$-indexed models. Hence, we deduce by the equivalence

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{G}^{\mathbb{T}}) \simeq \mathbf{Sh}(\mathcal{FG}(\mathbb{N}))$$

the representation result in [37, Theorem 5.1] (see also Corollary VII.56) in the particular case where $\mathbb{T}$ is a countable theory.

Remark VI.23. For a countable theory $\mathbb{T}$, the representing topological groupoid for $\mathcal{E}_{\mathbb{T}}$ constructed by Butz and Moerdijk in [17] (see also Definitions VII.54) is not directly comparable with the syntactic groupoid $\mathbb{G}^{\mathbb{T}}$, instead deriving from one of the many other open covers of $\mathcal{E}_{\mathbb{T}}$. In summary, it is the groupoid obtained when, instead of considering the theory $\mathcal{PQ}_{\mathbb{N}}$ of partial equivalence relations on $\mathbb{N}$ as we did, one takes the theory of partial equivalence relations on $\mathbb{N}$ where every equivalence class is infinite – that is, the theory obtained by adding to $\mathcal{PQ}_{\mathbb{N}}$, for each $n, l \in \mathbb{N}$, the axiom

$$[n \sim n] \vdash \bigvee \{ [n \sim m_1] \wedge \cdots \wedge [n \sim m_l] \mid m_i \in \mathbb{N} \text{ with } m_1 < m_2 < \cdots < m_l \}$$

expressing that the equivalence class of n in the subquotient has at least l many elements (and hence infinitely many).

Chapter VII

Topological representing groupoids

Topological representation for predicate theories. Logical theories can be 'represented' by topological structures. Let $\mathbb{T}$ be a propositional theory with a classifying locale $L_{\mathbb{T}}$. We intuit that $\mathbb{T}$ is *represented* by a set of models X_0, by which we mean that $L_{\mathbb{T}}$ is the frame of opens on some topological space whose points are X_0, if and only if the models X_0 are jointly conservative.

In this chapter we demonstrate a first-order generalisation of this observation. Propositional theories are replaced by predicate theories, classifying locales are replaced by classifying topoi, and topological spaces – inspired by the representation results of Joyal and Tierney [68] and Butz and Moerdijk [17] – are replaced by *open* topological groupoids. Thus, rather than representing the predicate theory as a topological space, we represent the theory by 'a space where points have automorphisms'.

Therefore, we will say that a theory is *represented* by an open topological groupoid if its topos of sheaves classifies the theory.

The classification result. The representation result of Butz and Moerdijk [17] expresses that a geometric theory admits a representation by an open topological groupoid if and only if the set-based models are a jointly conservative class of models. Our classification result answers the next obvious question: which open topological groupoids represent a given geometric theory? Informally, this question is equivalent to asking: which groupoids of models 'have enough information' to recover the theory?

The main result of this chapter is a characterisation of when a groupoid of models of a (geometric) theory can be endowed with topologies to yield a representing open topological groupoid.

We will observe that, unlike for propositional theories, it no longer suffices to simply have a groupoid of jointly conservative models. Instead, a further, model-theoretic condition, *elimination of parameters*, must be placed on the groupoid. Taken together, these conditions yield the characterisation of representing groupoids. In addition to admitting novel applications, our characterisation also subsumes the previous examples of representing groupoids found in the literature.

Representing groupoids for doctrinal sites. Recall that one of the intended applications of the geometric completion developed in Chapter IV is to replace an *ad hoc* approach to the model theory for logical theories from diverse syntaxes with a single unified approach using geometric logic. Therefore, during this chapter, we will assume that our theory is a geometric theory and argue in the familiar language of geometric logic.

Thus, by characterising the possible representing groupoids of a geometric theory, we have also characterised the possible representing groupoids of any predicate theory with a classifying topos, since such a theory must be *semantically equivalent* to a geometric one. In particular, this classification can be phrased in the language of doctrinal sites developed in Chapter III and Chapter IV to abstractly represent predicate theories without prejudice as to the underlying syntax.

Relation to the previous literature. The previous literature on using groupoids to represent topoi can be divided as to whether *localic* (i.e. pointfree) or topological groupoids are used. Both approaches have markedly different flavours.

By Caramello's *topological Galois theory* [21], a complete and atomic theory is represented by the topological group of automorphisms of a model if and only if that model is ultrahomogeneous. This is in contrast to the *localic Galois theory* developed by Joyal, Tierney and Dubuc ([68, Theorem VIII.3.1] and [34]), wherein it is shown that a complete and atomic theory is represented by the localic automorphism group of *any* model. We take this as evidence that while the disciplines of categorical logic and classical model theory, in which ultrahomogeneous models play an important role, are normally viewed as entirely distinct, this is not the case when we prioritise a topological rather than localic viewpoint.

In a similar fashion, Blass and Ščedrov characterise Boolean coherent topoi in [11] as those topoi that can be expressed as the coproduct of topoi of continuous actions by *coherent* topological groups. Moreover, these groups can be taken as the automorphism groups of ultrahomogeneous models for the theory classified by the topos.

In Chapter VI, we reviewed the celebrated result of Joyal and Tierney [68] that every topos is the topos of sheaves on some open localic groupoid. The parallel topological result was given in [17], where Butz and Moerdijk show that every topos with enough points is represented by an open topological groupoid. When a topos with enough points is known to classify a theory $\mathbb{T}$, Forssell's thesis and subsequent papers with Awodey [5], [36], [37] give an explicitly logical description of a representing open topological groupoid. Namely, their results express that $\mathbb{T}$ is represented by the groupoid of all $\mathfrak{K}$-*indexed models*, for a sufficiently large cardinal $\mathfrak{K}$ (we shall call such groupoids *Forssell groupoids*, see also Definition VII.53 and Section VI.3.3).

In summary, the relevant literature on the representation of topoi by localic and topological groupoids can be divided as follows.

	Localic representation	Topological representation
Connected atomic topoi	localic groups [34], [68],	topological groups [21],
Boolean coherent topoi		coproduct of coherent topological groups [11],
All topoi (with enough points)	open localic groupoids [68],	open topological groupoids [5], [17], [36], [37].

Our characterisation of the representing open topological groupoids recovers the previous results for the right-hand column of the above table.

Overview. The chapter proceeds as follows.

(A) Section VII.1 is divided into four parts. In the former two, Section VII.1.1 and Section VII.1.2, we define indexings of sets of models and an extension of the notion of a definable subset of a model to indexed groupoids of models. This will allow us to express the statement of our classification theorem for representing open topological groupoids, completed in Section VII.1.3. Finally, the method we will follow in proving the classification result, completed in Sections VII.2 to VII.4, is laid out in Section VII.1.4. Section VII.1.4 also includes a brief discussion of the relation between our result and the descent theory of Joyal and Tierney [68] recalled in Section VI.2.

(B) The proof of our classification result is contained in Sections VII.2 to VII.4. Given a groupoid of models $\mathbb{X} = (X_1 \rightrightarrows X_0)$ of our theory, the possible topologies with which X_0, respectively X_1, can be endowed that lead to a representing open topological groupoid are characterised in Section VII.2, resp., Section VII.3. The final steps of the proof of the classification result are completed in Section VII.4.

(C) In Section VII.5, we present some applications of our characterisation, including a demonstration that the other logical treatments of representing open topological groupoids considered in the literature can be recovered via our classification result.

   (a) In Section VII.5.1, we recover the principal result of [21] that an atomic theory is represented by the automorphism group of a single model if and only if that model is conservative and ultrahomogeneous, as well as a characterisation of Boolean topoi with enough points that is reminiscent of [11].
   (b) Section VII.5.2 concerns the representing groupoids of decidable theories.
   (c) In Section VII.5.3 we show that every open representing model groupoid is Morita equivalent to its *étale completion*.
   (d) The representation results of Awodey, Butz, Forssell and Moerdijk [5], [17], [37], including the case of Forssell groupoids, are recovered in Section VII.5.4.
   (e) Having described representing groupoids for a given theory, we answer in Section VII.5.5 the converse problem by adapting the methods of [52, Theorem 4.14] to describe a theory represented by a given groupoid of indexed structures.

(D) As a demonstration, in Section VII.6 we give a worked example in further detail of a representing groupoid for the theory of algebraic integers.

(E) Finally, Section VII.7 contains a translation of our classification result for geometric theories into the language of doctrinal sites. Recall that these were used in Part A to abstractly represent formal systems of predicate reasoning. Thereby, we obtain a classification of the representing open topological groupoids of any predicate theory with a classifying topos.

VII.1 The classification theorem

In order to state the classification theorem for representing open topological groupoids, we must first develop our terminology for *indexed structures* and *definables*. The former notion is jointly inspired the signature of the *diagram* of a model (see [89, Definition 2.3.2]) and the *enumerated models* and *indexed models* studied in, respectively, [17] and [5], [36], [37] (the connection with these works will be fully illustrated in Section VII.5.4). Indexed structures capture the intuition of constructing models from a list of parameter names. Meanwhile, the latter notion of definables extends the standard notion of definable subset found in model theory (see [52, §3]). The two properties that characterise representing open topological groupoids, 'conservativity' and 'elimination of parameters', are introduced in Section VII.1.3, in which we also state the classification theorem. 'Conservativity' will be a familiar notion to the logician, but we believe 'elimination of parameters' to be a novel addition.

VII.1.1 Indexed structures

Let Σ be a signature. Given a Σ-structure M, a standard model-theoretic construction is to consider M as a structure over the expanded signature $\Sigma \cup \{ c_n \mid n \in M \}$, the signature of the *diagram* of M, where we have added a constant symbol for each element of M. This allows us to express via formulae over the expanded signature those subsets of M that are defined in relation to finite tuples of other elements of M. We present a modification of this construction below.

Definition VII.1. Let Σ be a signature with N sorts, and let $\mathfrak{K} = (\mathfrak{K}_k)_{k \in N}$ be an N-tuple of sets. We denote by $\Sigma \cup \mathfrak{K}$ the expanded signature obtained by adding, for each $m \in \mathfrak{K}_k$, a constant symbol c_m (of the kth sort) to Σ. We will call these added constant symbols *parameters*.

A $\mathfrak{K}$-*indexing* of a Σ-structure M consists of:

(i) a *sub-expansion* of $\Sigma \cup \mathfrak{K}$, that is the signature

$$\Sigma \cup \left\{ c_m \,\middle|\, m \in \mathfrak{K}'_k,\ k \in N \right\}$$

for a tuple $\mathfrak{K}' = (\mathfrak{K}'_k)_{k \in N}$ of subsets $\mathfrak{K}'_k \subseteq \mathfrak{K}_k$,

(ii) and an interpretation of M as a structure over the signature

$$\Sigma \cup \left\{ c_m \,\middle|\, m \in \mathfrak{K}'_k,\ k \in N \right\}$$

such that, for each $k \in N$, the model M satisfies the sequent

$$\top \vdash_x \bigvee_{m \in \mathfrak{K}'_k} x = c_m.$$

In other words, we have interpreted in M some of the parameters introduced by $\mathfrak{K}$ in such a way that every element $n \in M$ is the interpretation of a parameter.

Our definition of $\mathfrak{K}$-indexed structure is equivalent to the homonymous notion found in [5], [36], [37] that a Σ-structure is $\mathfrak{K}$-indexed if the interpretation of the kth sort M^{A_k} is presented as a subquotient of $\mathfrak{K}_k$, i.e.

(i) there is a partial surjection $\mathfrak{K}_k \twoheadrightarrow M^{A_k}$,

(ii) or equivalently, there is a subset $S \subseteq \mathfrak{K}_k$ and an equivalence relation $\sim$ on S such that $M^{A_k} = S/\sim$.

We will abuse notation and write m for both the parameter as an element of $\mathfrak{K}$ and its interpretation in an $\mathfrak{K}$-indexed structure M. We denote a choice of $\mathfrak{K}$-indexing of M by $\mathfrak{K} \twoheadrightarrow M$.

Definition VII.2. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of models of $\mathbb{T}$ over a signature Σ. An $\mathfrak{K}$-*indexing* of $\mathbb{X}$ is a $\mathfrak{K}$-indexing $\mathfrak{K} \twoheadrightarrow M$ for each model $M \in X_0$.

Note that an element $n \in M$ can be the interpretation of multiple parameters, and also that the models $M \in X_0$ are allowed to share parameters, i.e. for $M, N \in X_0$, the same parameter $m \in \mathfrak{K}$ can be interpreted in both M and N.

Examples VII.3. Every (small) groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ of $\mathbb{T}$-models admits multiple indexings by various sets of parameters.

(i) Every model is trivially indexed by its own elements, and so $\mathbb{X}$ can be indexed by the set $\bigcup_{M \in X_0} M$.

(ii) Since $\mathbb{X}$ is a small groupoid, there is a sufficiently large cardinal $\mathfrak{K}$ such that every $M \in X_0$ is of cardinality at most $\mathfrak{K}$, and thus there is a choice of partial surjection $\mathfrak{K} \twoheadrightarrow M$ for each $M \in X_0$.

VII.1.2 Definables

In this subsection, we generalise definable subsets of a single model to groupoids of models, and show that this generalisation naturally carries the structure of an equivariant sheaf over the groupoid of models in question (when endowed with the discrete topologies).

Let M be a model of a theory $\mathbb{T}$. We use the notation $[\![\vec{x} : \varphi]\!]_M$ to denote the subset *defined* by the formula in context $\{ \vec{x} : \varphi \}$, i.e.

$$[\![\vec{x} : \varphi]\!]_M = \{ \vec{n} \in M \mid M \vDash \varphi(\vec{n}) \}.$$

The notation $\varphi(M)$ is also standard for definable subsets. We maintain reference to the context since we wish to emphasise the difference between the same formula interpreted in different contexts, for example

$$[\![\emptyset : \top]\!]_M = 1 \neq [\![x : \top]\!]_M$$

(assuming that M has more than one element).

Definitions VII.4. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of models of a theory $\mathbb{T}$.

(i) The *definable* of a formula in context $\{ \vec{x} : \varphi \}$, which we denote by $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$, is the coproduct

$$\coprod_{M \in X_0} [\![\vec{x} : \varphi]\!]_M.$$

Elements of $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ we denote as pairs $\langle \vec{n}, M \rangle$, where $\vec{n} \subseteq M \in X_0$.

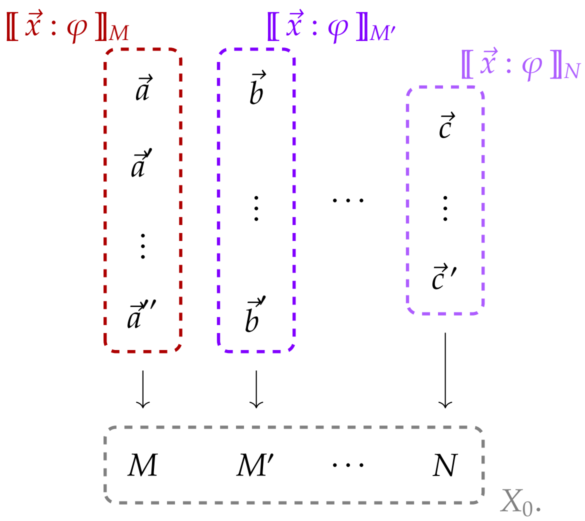
(ii) Suppose the models $M \in X_0$ are indexed by a set of parameters $\mathfrak{K}$. For a tuple $\vec{m}$ of parameters of $\mathfrak{K}$, we denote by $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ the *definable with parameters*

$$[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}} = \coprod_{M \in X_0} \{\, \langle \vec{n}, M \rangle \mid \vec{m} \in M \text{ and } M \vDash \psi(\vec{n}, \vec{m}) \,\}.$$

Recall that our models may share parameters, and so the definable $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ can have as elements $\langle \vec{n}, M \rangle, \left\langle \vec{n}', N \right\rangle$, where M and N differ.

If $\vec{m} = \emptyset$, we say that the definable $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ is *definable without parameters*. Note that every definable with parameters $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ is equivalently a definable without parameters over the expanded signature $\Sigma \cup \mathfrak{K}$.

A definable $[\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}$ possesses an evident projection $\pi_{[\![\, \vec{x}:\varphi \,]\!]}$ to X_0 which sends the pair $\langle \vec{n}, M \rangle$ to the model $M \in X_0$, as visually represented in the bundle diagram



Functoriality of definables. Note that the definable subset $[\![\, x : \top \,]\!]_M$ of a single model M is just the interpretation of the sort of the variable x, which we will denote by M^x. Similarly, $[\![\, \vec{x} : \top \,]\!]_M$ is the set of all tuples with the same sort as $\vec{x}$, or the product set $\prod_{x_i \in \vec{x}} M^{x_i}$, which we will denote by $M^{\vec{x}}$. For every formula φ in context $\vec{x}$, there is clearly an inclusion

$$[\![\, \vec{x} : \varphi \,]\!]_M \subseteq [\![\, \vec{x} : \top \,]\!]_M = M^{\vec{x}},$$

and similarly there is an inclusion $[\![\, \vec{x} : \varphi \,]\!]_M \subseteq [\![\, \vec{x} : \psi \,]\!]_M$ if $\mathbb{T}$ proves the sequent $\varphi \vdash_{\vec{x}} \psi$.

Let $\sigma \colon \vec{y} \to \vec{x}$ be a relabelling of variables, i.e. a map where y_i and $\sigma(y_i)$ have the same sort for each $y_i \in \vec{y}$. The map σ induces universally an arrow

$$\begin{tikzcd}
M^{\vec{x}} \arrow[r, dashed, "\sigma^M"] \arrow[d, "\mathrm{pr}_{\sigma(y_i)}"'] & \prod_{y_i \in \vec{y}} M^{y_i} = M^{\vec{y}} \arrow[d, "\mathrm{pr}_{y_i}"] \\
M^{\sigma(y_i)} \arrow[r, equal] & M^{y_i}.
\end{tikzcd}$$

If $\mathbb{T}$ proves the sequent $\varphi \vdash_{\vec{y}} \psi[\vec{x}/_{\sigma}\vec{y}]$, i.e. if there is an arrow $(\, \vec{x}, \varphi \,) \xrightarrow{\sigma} (\, \vec{y}, \psi \,)$ in the category $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$ from Chapter III, then the map σ^M restricts to a function on the subsets

$$\begin{tikzcd}
\llbracket \vec{x} : \varphi \rrbracket_M \arrow[r, dashed] \arrow[d, tail] & \llbracket \vec{y} : \psi \rrbracket_M \arrow[d, tail] \\
M^{\vec{x}} \arrow[r, "\sigma^M"] & M^{\vec{y}}.
\end{tikzcd}$$

We thus obtain a functor $\llbracket - \rrbracket_M \colon \mathbf{Con}_N \rtimes F^{\mathbb{T}} \to \mathbf{Sets}$. In fact, under the equivalences

$$\mathbb{T}\text{-mod}(\mathbf{Sets}) \simeq \mathbf{Geom}(\mathbf{Sets}, \mathcal{E}_{\mathbb{T}}) \simeq K_{F^{\mathbb{T}}}\text{-}\mathbf{Flat}(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, \mathbf{Sets}),$$

this is precisely the continuous flat functor corresponding to the $\mathbb{T}$-model M in **Sets**.

Evidently, the pointwise coproduct $\coprod_{M \in X_0} \llbracket - \rrbracket_M$ behaves well with the projection $\pi_{\llbracket - \rrbracket}$ in that, for every arrow $(\vec{y}, \varphi) \xrightarrow{\sigma} (\vec{x}, \psi)$ of $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$, the diagram

$$\begin{tikzcd}
\llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \arrow[rr] \arrow[d, tail] \arrow[ddr, bend right=60, "\pi_{\llbracket \vec{x}:\varphi \rrbracket}"'] & & \llbracket \vec{y} : \psi \rrbracket_{\mathbb{X}} \arrow[d, tail] \arrow[ddl, bend left=60, "\pi_{\llbracket \vec{y}:\psi \rrbracket}"] \\
\coprod\limits_{M \in X_0} M^{\vec{x}} \arrow[rr, "\coprod_{M \in X_0} \sigma^M"] \arrow[dr] & & \coprod\limits_{M \in X_0} M^{\vec{y}} \arrow[dl] \\
& X_0 &
\end{tikzcd}$$

commutes. Thus, we obtain a functor

$$\llbracket - \rrbracket_{X_0} \colon \mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_0 \simeq \mathbf{Sets}^{X_0}.$$

Actions on definables. The bundle $\pi_{\llbracket \vec{x}:\varphi \rrbracket} \colon \llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \to X_0$ admits a canonical *lifting* of the X_1-action on X_0. By this we mean there is a map

$$\theta_{\llbracket \vec{x}:\varphi \rrbracket} \colon \llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \times_{X_0} X_1 \to \llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}},$$

where $\llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \times_{X_0} X_1$ is the pullback

$$\begin{tikzcd}
\llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \times_{X_0} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}} \arrow[d, "\pi_{\llbracket \vec{x}:\varphi \rrbracket}"] \\
X_1 \arrow[r, "s"] & X_0,
\end{tikzcd}$$

satisfying the equations

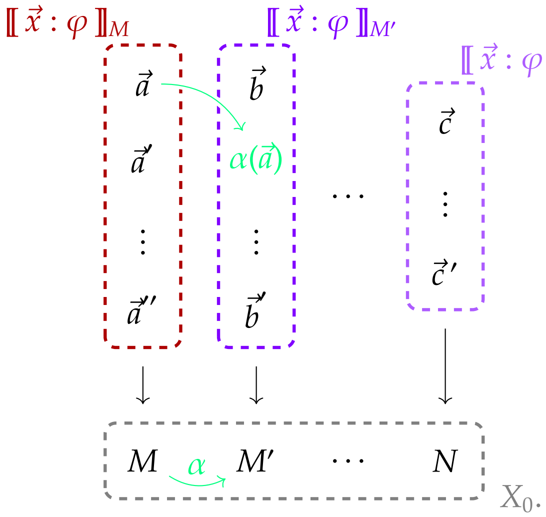
$$\theta_{\llbracket \vec{x}:\varphi \rrbracket}(\theta_{\llbracket \vec{x}:\varphi \rrbracket}(\langle \vec{m}, M \rangle, \alpha), \gamma) = \theta_{\llbracket \vec{x}:\varphi \rrbracket}(\vec{m}, \gamma \circ \alpha),$$
$$\theta_{\llbracket \vec{x}:\varphi \rrbracket}(\langle \vec{m}, M \rangle, \mathrm{id}_M) = \langle \vec{m}, M \rangle,$$

i.e. $(\llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}}, \pi_{\llbracket \vec{x}:\varphi \rrbracket}, \theta_{\llbracket \vec{x}:\varphi \rrbracket})$ is a sheaf on the groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$, in the sense of Definition V.2(i), when X_1 and X_0 are both endowed with the discrete topology.

Given a $\mathbb{T}$-model isomorphism $M \xrightarrow{\alpha} M' \in X_1$, we declare that

$$\theta_{[\![\vec{x}:\varphi]\!]}(\langle \vec{a}, M\rangle, \alpha) = \langle \alpha(\vec{a}), M'\rangle,$$

where $\langle \vec{a}, M\rangle \in [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$. Since α is a morphism of $\mathbb{T}$-models, it preserves the interpretation of the formula φ, i.e. $M' \vDash \varphi(\alpha(\vec{a}))$ too. Hence, $\theta_{[\![\vec{x}:\varphi]\!]}$ is well-defined. The action $\theta_{[\![\vec{x}:\varphi]\!]}$ can be visualised as acting on the bundle $\pi_{[\![\vec{x}:\varphi]\!]} \colon [\![\vec{x} : \varphi]\!]_{\mathbb{X}} \to X_0$ as in the diagram



For each arrow $(\vec{y}, \varphi) \xrightarrow{\sigma} (\vec{x}, \psi)$ of $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$, it can be checked directly that the induced map $[\![\sigma]\!]_{\mathbb{X}} \colon [\![\vec{x} : \varphi]\!]_{\mathbb{X}} \to [\![\vec{y} : \psi]\!]_{\mathbb{X}}$ is equivariant with respect to the above defined X_1-actions, i.e. the square

$$\begin{tikzcd}
{[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \times_{X_0} X_1} \arrow[r, "\theta_{[\![\vec{x}:\varphi]\!]}"] \arrow[d, "{[\![\sigma]\!]_{\mathbb{X}}}"] & {[\![\vec{x} : \varphi]\!]} \arrow[d, "{[\![\sigma]\!]_{\mathbb{X}}}"] \\
{[\![\vec{y} : \psi]\!] \times_{X_0} X_1} \arrow[r, "\theta_{[\![\vec{y}:\psi]\!]}"] & {[\![\vec{y} : \psi]\!]}
\end{tikzcd}$$

commutes. Thus, there is a functor $[\![-]\!]_{\mathbb{X}} \colon \mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}} \to \mathbf{Sets}^{\mathbb{X}}$.

Orbits of definables. Note that every definable with parameters forms a subset of a definable without parameters, e.g.

$$[\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \exists \vec{y}\, \varphi]\!]_{\mathbb{X}}, [\![\vec{x} : \top]\!]_{\mathbb{X}}.$$

The X_1-action $\theta_{[\![\vec{x}:\top]\!]}$ does not restrict to an X_1-action on $[\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}}$ since the subset may not be closed under the action, i.e. if $\langle \vec{a}, M\rangle \in [\![\vec{x}, \vec{m} : \varphi]\!]$ and there is an isomorphism of $\mathbb{T}$-models $M \xrightarrow{\alpha} M' \in X_1$, it does not follow that $M' \vDash \varphi(\alpha(\vec{a}), \vec{m})$. This is because α is an isomorphism only of the Σ-structure and need not preserve the interpretation of any of the parameters we have added. Indeed, $\vec{m}$ might not even be interpreted in M'. Of course, if $\vec{m} = \emptyset$, then $[\![\vec{x} : \varphi]\!]$ is closed under the X_1-action $\theta_{[\![\vec{x}:\top]\!]}$, the restricted action being precisely $\theta_{[\![\vec{x}:\varphi]\!]}$.

Definition VII.5. The *orbit* $\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}}$ of a definable with parameters $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ is the closure under the isomorphisms contained in X_1, explicitly

$$\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}} = \left\{ \langle \vec{n}, N \rangle \,\middle|\, \exists M \xrightarrow{\alpha} N \in X_1 \text{ such that } M \vDash \psi(\alpha^{-1}(\vec{n}), \vec{m}) \right\}.$$

Equivalently, $\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}}$ defines the smallest stable subset of $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$ that contains $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$.

VII.1.3 Statement of the classification theorem

Having developed our terminology for the definables of a groupoid of $\mathbb{T}$-models indexed by $\mathfrak{K}$, we can now state the classification theorem.

Definitions VII.6. Let $\mathbb{T}$ be a geometric theory over a signature Σ and let $\mathbb{X}$ be a (small) groupoid of models of $\mathbb{T}$ indexed by a set of parameters $\mathfrak{K}$.

(i) We say that $\mathbb{X}$ is *conservative* if X_0 is a conservative set of models, i.e. for each pair of geometric formulae over Σ in context $\vec{x}$, if $[\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}} \subseteq [\![\, \vec{x} : \psi \,]\!]_{\mathbb{X}}$, then $\mathbb{T}$ proves the sequent $\varphi \vdash_{\vec{x}} \psi$.

(ii) We say that $\mathbb{X}$ *eliminates parameters* if the orbit of each definable with parameters $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ is definable without parameters, i.e. there exists a geometric formula in context $\{\, \vec{x} : \varphi \,\}$ such that

$$\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}} = [\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}.$$

Remarks VII.7. (i) Recall that a topos $\mathcal{E}$ *has enough points* if the points, i.e. the geometric morphisms **Sets** $\to \mathcal{E}$, are jointly conservative – that is the inverse image functors are jointly faithful. If $\mathcal{E}$ classifies a theory $\mathbb{T}$, then $\mathcal{E}$ has enough points if and only if the set-based models of $\mathbb{T}$ are conservative. By [57, Corollary 7.17], if $\mathcal{E}$ has enough points, a *small* set of conservative models can always be chosen. We will therefore mix our terminology and say that a theory has enough points to mean there exists a conservative set of set-based models.

(ii) Our terminology 'elimination of parameters' is justified since, in the special case of field theory, it is demonstrably the groupoid removing the parameters from the defining polynomial of a solution set. As a simple example from traditional Galois theory, the orbit of the definable with parameters $[\![\, x = i \,]\!]_{\mathbb{Q}(i)} = \{\, i \,\}$ under the automorphisms of $\mathbb{Q}(i)$ that fix $\mathbb{Q}$ is definable without parameters, namely

$$\overline{[\![\, x = i \,]\!]_{\mathrm{Aut}(\mathbb{Q}(i))}} = \{\, i, -i \,\} = [\![\, x^2 = -1 \,]\!]_{\mathrm{Aut}(\mathbb{Q}(i))}.$$

Our terminology 'elimination of parameters' is also inspired by the parallel model-theoretic study of Galois theory and the theory of elimination of imaginaries of Bruno Poizat [101, §2] that arises therein.

(iii) Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of models of $\mathbb{T}$ indexed by a set of parameters $\mathfrak{K}$. To check that the groupoid $\mathbb{X}$ eliminates parameters, it suffices to show that, for each tuple of parameters $\vec{m}$, there exists a formula in context $\{\, \vec{y} : \chi \,\}$ without parameters such that

$$\overline{[\![\, \vec{y} = \vec{m} \,]\!]_{\mathbb{X}}} = [\![\, \vec{y} : \chi \,]\!]_{\mathbb{X}}$$

since, for an arbitrary definable with parameters $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$, we have that

$$\begin{aligned} \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} &= \overline{[\![\vec{x} : \exists \vec{y}\, \psi[\vec{y}/\vec{m}] \wedge \vec{y} = \vec{m}]\!]_{\mathbb{X}}}, \\ &= [\![\vec{x} : \exists \vec{y}\, \psi[\vec{y}/\vec{m}] \wedge \chi]\!]_{\mathbb{X}}. \end{aligned}$$

(iv) Let $\mathbb{T}$ be a geometric theory over a signature Σ and let $\mathbb{X}$ be a groupoid of $\mathbb{T}$-models indexed by $\mathfrak{K}$. We note that the condition that $\mathbb{X}$ eliminates parameters does not depend on the theory $\mathbb{T}$, but only on the signature Σ. We will revisit this observation in Remark VII.30.

(v) Let $\mathbb{X}$ be a conservative groupoid of $\mathbb{T}$-models, indexed by a set of parameters $\mathfrak{K}$ for which $\mathbb{X}$ eliminates parameters. Given a definable formula with parameters $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$, we may wonder what restrictions can be made on the formula φ which witnesses the elimination of parameters, i.e. a formula for which $\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$. We note that, since

$$[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}} \subseteq \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} \subseteq [\![\vec{x} : \exists \vec{y}\, \psi]\!]_{\mathbb{X}}$$

and the groupoid $\mathbb{X}$ is conservative, $\mathbb{T}$ must prove the sequent $\varphi \vdash_{\vec{x}} \exists \vec{y}\, \psi$. Similarly, if every instance of the parameters $\vec{m}$ in a model $M \in X_0$ satisfies a formula χ, then $\mathbb{T}$ also proves the sequent $\varphi \vdash_{\vec{x}} \exists \vec{y}\, \psi \wedge \chi$. In particular, we have that

$$\varphi \vdash_{\vec{x}} \exists \vec{y}\, \psi \wedge \bigwedge_{m_i = m_j} y_i = y_j,$$

where the conjunction $\bigwedge_{m_i=m_j} y_i = y_j$ ranges over the elements of the tuple $\vec{m}$ that are equal.

Theorem VII.8 (Classification of representing open topological groupoids for a geometric theory)**.** *Let $\mathbb{T}$ be a geometric theory, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of set-based models of $\mathbb{T}$. The following are equivalent.*

(i) There exist topologies on X_0 and X_1 making $\mathbb{X}$ an open topological groupoid such that there is an equivalence of topoi $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$.

(ii) The groupoid $\mathbb{X}$ is conservative and there exists a set of parameters $\mathfrak{K}$ and an indexing of $\mathbb{X}$ by $\mathfrak{K}$ for which $\mathbb{X}$ eliminates parameters.

Remark VII.9. Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X}$ be a groupoid of models for $\mathbb{T}$ satisfying the hypotheses of Theorem VII.8.

(i) In general, we can not *a priori* infer an indexing for which $\mathbb{X}$ eliminates parameters without knowledge of the topologies on $\mathbb{X}$ for which $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$. In other words, there does not exist a canonical indexing $\mathfrak{K}_{\text{can}} \rightharpoonup \mathbb{X}$ with the property that there exist topologies making $\mathbb{X}$ an open topological groupoid with $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$ if and only if $\mathbb{X}$ eliminates parameters for the canonical indexing $\mathfrak{K}_{\text{can}} \rightharpoonup \mathbb{X}$. This is because, as will become apparent in Section VII.2, a choice of indexing for $\mathbb{X}$ yields a choice of topology on X_0 and, vice versa, a choice of topology on X_0 yields a choice of indexing for $\mathbb{X}$.

(ii) If $\mathbb{X}'$ is another groupoid for which there is an equivalence of categories $\mathbb{X} \simeq \mathbb{X}'$, then it is not necessarily true that $\mathbb{X}'$ can be endowed with topologies for which $\mathbf{Sh}(\mathbb{X}') \simeq \mathcal{E}_{\mathbb{T}}$. This apparent defect arises because we are considering topological categories. Indeed, if $\mathbb{X}'$ and $\mathbb{X}$ were equivalent, or *homeomorphic*, as *topological categories*, by which we mean that the functors $F \colon \mathbb{X} \to \mathbb{X}'$ and $G \colon \mathbb{X}' \to \mathbb{X}$ that witness the equivalence also preserve the induced topological structure, then there would be an equivalence of topoi $\mathbf{Sh}(\mathbb{X}') \simeq \mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$.

The proof of Theorem VII.8 is completed in Sections VII.2 to VII.4. Given a geometric theory $\mathbb{T}$, we will call a (small) groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ of set-based $\mathbb{T}$-models, i.e. a small subcategory $\mathbb{X} \subseteq \mathbb{T}\text{-mod}(\mathbf{Sets})$ in which every arrow has an inverse, a *model groupoid* for $\mathbb{T}$.

VII.1.4 Our method

We now lay out the method that we will follow in Sections VII.2 to VII.4 in order to prove Theorem VII.8. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a groupoid and let

$$\mathbf{p} \colon \mathbf{Sets}^{\mathbb{X}} \simeq \mathbf{Sh}(\mathbb{X}^{\delta}_{\delta}) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

be a geometric morphism. For each object $x \in X_0$ of $\mathbb{X}$, the evaluation functor $\mathbf{Sets}^{\mathbb{X}} \to \mathbf{Sets}$,

$$[P \colon \mathbb{X} \to \mathbf{Sets}] \mapsto P(x)$$

yields a geometric morphism $\mathrm{ev}_x \colon \mathbf{Sets} \to \mathbf{Sets}^{\mathbb{X}}$, and hence a point

$$\mathbf{Sets} \xrightarrow{\mathrm{ev}_x} \mathbf{Sets}^{\mathbb{X}} \xrightarrow{\mathbf{p}} \mathcal{E}_{\mathbb{T}}$$

of the topos $\mathcal{E}_{\mathbb{T}}$. Consequently, $x \in X_0$ also yields a $\mathbb{T}$-model M.

Similarly, every isomorphism $x \xrightarrow{\alpha} y$ of X_1 yields an isomorphism of points

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, bend left=50, "\mathrm{ev}_x"{name=U}] \arrow[r, bend right=50, "\mathrm{ev}_y"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & \mathbf{Sets}^{\mathbb{X}} \arrow[r, "\mathbf{p}"] & \mathcal{E}_{\mathbb{T}} \end{tikzcd}$$

and thefore an isomorphism of $\mathbb{T}$-models. Thus, the geometric morphism $\mathbf{p}$ corresponds to a model groupoid $\mathbb{X} \subseteq \mathbb{T}\text{-mod}(\mathbf{Sets})$. Similarly, each model groupoid $\mathbb{X} \subseteq \mathbb{T}\text{-mod}(\mathbf{Sets})$ also yields a geometric morphism

$$\mathbf{p} \colon \mathbf{Sets}^{\mathbb{X}} \longrightarrow \mathcal{E}_{\mathbb{T}}.$$

Recalling that $(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}})$ is a site of definition for $\mathcal{E}_{\mathbb{T}}$, the continuous flat functor

$$\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}} \longrightarrow \mathbf{Sets}^{\mathbb{X}}$$

corresponding to the geometric morphism $\mathbf{p}\colon \mathbf{Sets}^{\mathbb{X}} \to \mathcal{E}_{\mathbb{T}}$ is precisely the functor $[\![-]\!]_{\mathbb{X}}$ considered in Section VII.1.2.

We will denote the composite

$$\mathbf{Sh}(X_0^\delta) \xrightarrow{v} \mathbf{Sh}(\mathbb{X}_\delta^\delta) \xrightarrow{\mathbf{p}} \mathcal{E}_{\mathbb{T}}$$

by $\mathbf{p}_0$.

Definition VII.10. A *factoring topology for objects* is a topology τ_0 on X_0 such that the geometric morphism $\mathbf{p}_0\colon \mathbf{Sh}(X_0^\delta) \to \mathcal{E}$ factors as

$$\mathbf{Sh}(X_0^\delta) \xrightarrow{j} \mathbf{Sh}(X_0^{\tau_0}) \dashrightarrow \mathcal{E}_{\mathbb{T}}.$$

Given a factoring topology for objects τ_0, recall that, by Lemma V.16, there exists a unique (up to isomorphism) geometric morphism $\mathbf{p}'$ such that

$$\begin{tikzcd}
\mathbf{Sh}(X_0^\delta) \arrow[r, "j"] \arrow[d, "v"'] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d, "u^\delta"] \arrow[ddr, bend left] & \\
\mathbf{Sh}(\mathbb{X}_\delta^\delta) \arrow[r, "j'"] \arrow[ur, phantom, "\lrcorner"] \arrow[drr, bend right, "\mathbf{p}"'] & \mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta) \arrow[dr, dashed, "\mathbf{p}'"] & \\
 & & \mathcal{E}_{\mathbb{T}}
\end{tikzcd}$$

commutes.

Definition VII.11. Given a factoring topology for objects τ_0 on X_0, a *factoring topology for arrows* is a topology τ_1 on X_1 such that $\mathbb{X}_{\tau_0}^{\tau_1} = (X_1^{\tau_1} \rightrightarrows X_0^{\tau_0})$ is a topological groupoid and $\mathbf{p}'$ factors as

$$\mathbf{Sh}(\mathbb{X}_{\tau_0}^\delta) \xrightarrow{w} \mathbf{Sh}(\mathbb{X}_{\tau_0}^{\tau_1}) \dashrightarrow \mathcal{E}_{\mathbb{T}}.$$

Our method for proving Theorem VII.8 is broken down into four intermediate steps as follows.

(A) In Section VII.2 we classify, for a geometric theory $\mathbb{T}$ and a set X_0 of set-based models, the possible factoring topologies for X_0. We will show that, when the models in X_0 are indexed by some set of parameters $\mathfrak{K}$, the logical topology on objects, introduced in [36], [37], is a factoring topology for X_0 and that, up to a choice of indexing for each $M \in X_0$, every factoring topology for X_0 contains a logical topology on objects.

(B) Given a factoring topology on objects for X_0, we classify in Section VII.3 the factoring topologies on arrows for X_1. We will show that a topology τ_1 on X_1 is a factoring topology for arrows if and only if τ_1 contains the logical topology for arrows, another topology utilised in [36], [37] (where we have assumed that τ_0 contains the logical topology for objects for some indexing of the models).

(C) We demonstrate in Section VII.3.1 that, if X_0 and X_1 are endowed with the logical topologies $\tau\text{-log}_0$ and $\tau\text{-log}_1$, and that the resulting topological groupoid is *open*, then the factoring geometric morphism

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}} \tag{VII.i}$$

is localic.

(D) Finally in Section VII.4, we deduce Theorem VII.8 by studying the morphism of internal locales induced by the localic geometric morphism (VII.i) using the methods established in Chapter II.

Comparison with Joyal-Tierney descent. Before embarking on the main proof, we elaborate further the connection between the representation of topoi using topological groupoids and localic groupoids, as discussed in Chapter VI.

Let $\mathcal{E}$ be a topos. Let X_0 be a set of points of $\mathcal{E}$, i.e. $X_0 \subseteq \mathbf{Geom}(\mathbf{Sets}, \mathcal{E})$, and let τ_0 be a factoring topology for objects on X_0, i.e. there is a canonically induced geometric morphism

$$\mathbf{Sh}\left(X_0^{\tau_0}\right) \longrightarrow \mathcal{E}.$$

There are two groupoids that can naturally be associated with the pair (X_0, τ_0).

(i) Firstly, there is the *concrete* groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ (i.e. a groupoid internal to **Sets**) obtained by taking X_1 as the set of all isomorphisms of the points X_0 (which is always small).

(ii) Secondly, there is the *localic* groupoid $\mathbb{X}^{\text{loc}} = (X_1^{\text{loc}} \rightrightarrows X_0^{\text{loc}})$, whose locale of objects is the frame of opens $\mathcal{O}(X_0^{\tau_0})$ and whose locale of arrows is the locale in the (bi)pullback of topoi

$$\begin{tikzcd} \mathbf{Sh}(X_1^{\text{loc}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[d] \\ \mathbf{Sh}(X_0^{\tau_0}) \arrow[r] & \mathcal{E}. \end{tikzcd}$$

Suppose that there exists a topology τ_1 on X_1 making $\mathbb{X}^{\tau_1}_{\tau_0} = (X_1^{\tau_1} \rightrightarrows X_0^{\tau_0})$ an open topological groupoid for which there is an equivalence of topoi $\mathcal{E} \simeq \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$. Then, by Lemma V.8 and Lemma V.9, the canonical geometric morphism

$$u\colon \mathbf{Sh}\left(X_0^{\tau_0}\right) \longrightarrow \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \simeq \mathcal{E}.$$

is an open surjection, and so, by an application of the descent theory of Joyal and Tierney [68] (see also Theorem VI.5), the topos $\mathcal{E}$ is also the topos of sheaves on the localic groupoid $\mathbb{X}^{\text{loc}}$.

The converse, however, is not true. In Example VII.40 we give a counterexample consisting of a topos $\mathcal{E}$ and an open surjective point $p\colon \mathbf{Sets} \to \mathcal{E}$ for which there is an equivalence $\mathcal{E} \simeq \mathbf{B}\mathrm{Aut}(p)^{\text{loc}}$ (here $\mathrm{Aut}(p)^{\text{loc}}$ denotes the localic automorphism group of p) but where there is not an equivalence $\mathcal{E} \not\simeq \mathbf{B}\mathrm{Aut}(p)^{\tau_1}$ for any topology τ_1 on the concrete automorphism group $\mathrm{Aut}(p)$.

VII.2 Factoring topologies for objects

Let $\mathbb{T}$ be a geometric theory and X_0 a set of set-based models of $\mathbb{T}$. We wish to classify the possible factoring topologies for X_0. We first note that there is a factorisation

$$\begin{tikzcd} \mathbf{Sh}(X_0^\delta) \arrow[d, "j"'] \arrow[dr, "\mathbf{p}_0"] & \\ \mathbf{Sh}(X_0^{\tau_0}) \arrow[r, dashed] & \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}}) \end{tikzcd}$$

if and only if there is a factorisation

$$\begin{tikzcd} \mathbf{Sh}(X_0^\delta) & \\ \mathbf{Sh}(X_0^{\tau_0}) \arrow[u, "j"] & \mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}. \arrow[l, dashed] \arrow[ul, "\mathbf{p}_0^* \circ \ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}"'] \end{tikzcd}$$

In the following proposition, we restrict to the special case where $\mathbb{T}$ is a single-sorted theory for notational simplicity. However, a multi-sorted version is readily deduced using the same ideas – a statement is given in Remark VII.13.

Proposition VII.12. *Let $\mathbb{T}$ be a single-sorted geometric theory and let X_0 be a set of models of $\mathbb{T}$. A topology τ_0 on X_0 is a factoring topology for objects if and only if*

(i) there exists a topology T on $[\![x : \top]\!]_{\mathbb{X}} = \coprod_{M \in X_0} M$ such that the projection

$$\pi_{[\![x:\varphi]\!]} \colon [\![x : \top]\!]_{\mathbb{X}}^T \longrightarrow X_0^{\tau_0}$$

is a local homeomorphism,

(ii) and for each geometric formula in context $\{ \vec{x} : \varphi \}$, where the context $\vec{x}$ is of length n, the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in the product topology T^n on

$$[\![\vec{x} : \top]\!]_{\mathbb{X}} = \coprod_{M \in X_0} M^n = \left(\coprod_{M \in X_0} M \right)^n .$$

Proof. Suppose τ_0 is a factoring topology for objects, and let

$$k \colon \mathbf{Sh}(X_0^{\tau_0}) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

be the factoring geometric morphism. Hence, the map $\pi_{[\![x:\varphi]\!]} \colon [\![x : \top]\!]_{\mathbb{X}} \to X_0$, being the image $\mathbf{p}^* \circ \ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}} (x : \top)$, must be a local homeomorphism for some topology T on $[\![x : \top]\!]_{\mathbb{X}}$ when X_0 is endowed with the topology τ_0.

As $k^* \circ \ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}$ preserves finite limits we deduce that, for a context $\vec{x}$ of length n,

$$\begin{aligned} k^* \circ \ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\vec{x}, \top) &= k^* \circ \ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}((x : \top)^n), \\ &= (k^* \circ \ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(x : \top))^n, \\ &= \left[\pi_{[\![\vec{x}:\top]\!]} \colon [\![\vec{x} : \top]\!]_{\mathbb{X}}^{T^n} \to X_0^{\tau_0} \right]. \end{aligned}$$

The functor k^* also preserves subobjects. The subobjects of $\ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\vec{x}, \top)$ are objects of the form $\ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\vec{x}, \varphi)$ by Section II.2 (cf. also [63, Lemma D1.4.4]). Hence, since subobjects of $[\![\vec{x} : \top]\!]_{\mathbb{X}}^{T^n}$ are, in particular, open subsets, the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in T^n. This completes the 'only if' direction of the proof.

Conversely, suppose that:

(i) there is a topology T on $[\![x : \top]\!]_{\mathbb{X}}$ such that the projection $\pi_{[\![x:\varphi]\!]} \colon [\![x : \top]\!]_{\mathbb{X}}^{T} \to X_0^{\tau_0}$ is a local homeomorphism,

(ii) and for each geometric formula in context $\{ \vec{x} : \varphi \}$, the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in the product topology T^n on $[\![\vec{x} : \top]\!]_{\mathbb{X}}$.

Clearly, since finite products are computed in $\mathbf{Sh}(X_0^{\tau_0})$ as wide pullbacks in **Top**, $\pi_{[\![\vec{x}:\top]\!]} \colon [\![\vec{x} : \top]\!]_{\mathbb{X}}^{T^n} \to X_0^{\tau_0}$ is also a local homeomorphism, where T^n is the product topology, for each natural number n. As the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in the product topology T^n, the composite

$$k^* \circ \ell_{\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\vec{x}, \varphi) = \left[\pi_{[\![\vec{x}:\varphi]\!]} \colon [\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}} \to X_0^{\tau_0} \right]$$

is also a local homeomorphism, when $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ is endowed with the subspace topology.

Let $(\vec{x}, \varphi) \xrightarrow{\sigma} (\vec{y}, \psi)$ be a morphism of $\mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}}$, where the contexts $\vec{x}$ and $\vec{y}$ have respective lengths n and m. Since the induced map

$$\begin{tikzcd}
{[\![\vec{x} : \top]\!]_{\mathbb{X}} = \coprod_{M \in X_0} M^n} \arrow[rr, "{[\![\sigma]\!]_{\mathbb{X}}}"] \arrow[dr, "\pi_{[\![\vec{x}:\top]\!]}"'] & & {\coprod_{M \in X_0} M^m = [\![\vec{y} : \top]\!]_{\mathbb{X}}} \arrow[dl, "\pi_{[\![\vec{y}:\top]\!]}"] \\
& X_0 &
\end{tikzcd}$$

is obtained universally, it is automatically continuous when $[\![\vec{x} : \top]\!]$ and $[\![\vec{y} : \top]\!]$ are endowed with their respective product topologies. Therefore, the restriction to the subspaces

$$\begin{tikzcd}
{[\![\vec{x} : \varphi]\!]_{\mathbb{X}}} \arrow[rr, "{[\![\sigma]\!]_{\mathbb{X}}}"] \arrow[d, tail] & & {[\![\vec{y} : \psi]\!]_{\mathbb{X}}} \arrow[d, tail] \\
{[\![\vec{x} : \top]\!]_{\mathbb{X}}} \arrow[rr] \arrow[dr, "\pi_{[\![\vec{x}:\top]\!]}"'] & & {[\![\vec{y} : \top]\!]_{\mathbb{X}}} \arrow[dl, "\pi_{[\![\vec{y}:\top]\!]}"] \\
& X_0 &
\end{tikzcd}$$

is also a continuous map.

Thus, there exists a factorisation

$$\begin{tikzcd}
\mathbf{Sh}(X_0^{\delta}) & \\
\mathbf{Sh}(X_0^{\tau_0}) \arrow[u, "J"] & \mathbf{Con}_1^{\mathrm{op}} \rtimes F^{\mathbb{T}} \arrow[l, dashed] \arrow[ul, "\mathbf{p}_0^* \circ \ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}"']
\end{tikzcd}$$

and so τ_0 is a factoring topology. □

Remark VII.13. In Proposition VII.12, we assumed the geometric theory $\mathbb{T}$ was single-sorted for simplicity's sake. However, the proof is readily adapted to the multi-sorted case. We give the statement below.

Let $\mathbb{T}$ be an arbitrary geometric theory over a signature Σ, and let X_0 be a set of models of $\mathbb{T}$. A topology τ_0 on X_0 is a factoring topology for objects if and only if

(i) for each sort X in Σ, there exists a topology T_x on $[\![x : \top]\!]_{\mathbb{X}}$ such that the projection $\pi_{[\![x:\top]\!]} \colon [\![x : \top]\!]_{\mathbb{X}}^{T_x} \to X_0^{\tau_0}$ is a local homeomorphism, where x is a single variable of sort X,

(ii) for each geometric formula in context $\{ \vec{x} : \varphi \}$, the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in the corresponding product topology $T_{\vec{x}}$ on $[\![\vec{x} : \top]\!]_{\mathbb{X}}$.

The logical topology on objects. Given an indexing of each model in our set of objects by some parameters, we now introduce a certain class of factoring topologies for objects: the *logical topologies for objects*. This is an adaptation of the topology used in the papers by Awodey and Forssell (see [37, Definition 3.1]), itself an adaptation of the topology used by Butz and Moerdijk in [17, §2]. As we will see in Proposition VII.17, a factoring topology for objects τ_0 on X_0 can, essentially, be chosen to contain a logical topology.

Definition VII.14 (Definition 3.1 [37], Definition 1.2.1 [5])**.** Let $\mathbb{T}$ be a geometric theory and let X_0 be a set of $\mathbb{T}$-models indexed by a set of parameters $\mathfrak{K}$. The *logical topology for objects* $\tau\text{-log}_0$ on X_0, for this indexing, is the topology generated by the basis whose opens are *sentences with parameters* $[\![\vec{m} : \varphi]\!]_{\mathbb{X}}$, i.e. those definables with parameters $[\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}}$ where $\vec{x} = \emptyset$.

Remark VII.15. Let X_0 be a set of indexed $\mathbb{T}$-models as above. We note that, when generating the logical topology for objects $\tau\text{-log}_0$ on X_0, we can focus our attention on only those opens $[\![\vec{m} : \varphi]\!]_{\mathbb{X}}$ where φ is an *atomic formula* (see [63, Definition D1.1.3]). This is because any of the logical symbols $\{ \wedge, \bigvee, \exists \}$ used to construct a composite geometric formula from atomic ones can be replaced by topological constructions:

(i) $[\![\vec{m} : \varphi \wedge \psi]\!]_{\mathbb{X}} = [\![\vec{m} : \varphi]\!]_{\mathbb{X}} \cap [\![\vec{m} : \psi]\!]_{\mathbb{X}}$,

(ii) $[\![\vec{m} : \bigvee_{i \in I} \varphi_i]\!]_{\mathbb{X}} = \bigcup_{i \in I} [\![\vec{m} : \varphi_i]\!]_{\mathbb{X}}$,

(iii) $[\![\vec{m} : \exists y\, \varphi]\!]_{\mathbb{X}} = \bigcup_{\vec{m}' \in \mathfrak{K}} [\![\vec{m}, \vec{m}' : \varphi]\!]_{\mathbb{X}}$.

Lemma VII.16. *Let $\mathbb{T}$ be a single-sorted geometric theory. Each 'logical topology for objects' is a factoring topology for objects.*

Proof. Let X_0 be a set of models of $\mathbb{T}$ indexed by parameters $\mathfrak{K}$. By Proposition VII.12 and the multi-sorted version given in Remark VII.13, it suffices to show that, for each singleton variable, there is a topology T_x on $[\![x : \top]\!]_{\mathbb{X}}$ such that the projection

$$\pi_{[\![x:\top]\!]} \colon [\![x : \top]\!]_{\mathbb{X}} \longrightarrow X_0^{\tau\text{-log}_0}$$

is a local homeomorphism, and for each geometric formula in context $\{ \vec{x} : \varphi \}$ the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is open in the product topology $T_{\vec{x}}$ on $[\![\vec{x} : \top]\!]_{\mathbb{X}}$.

Let T_x be the topology on $[\![x : \top]\!]_{\mathbb{X}}$ generated by the basic opens

$$[\![x, \vec{m} : \varphi]\!]_{\mathbb{X}} = \{ \langle n, M \rangle \mid M \vDash \varphi(n, \vec{m}) \}.$$

For this topology, $\pi_{[\![x:\top]\!]}$ is a local homeomorphism. Firstly, each element $\langle n, M \rangle$ of $[\![x : \top]\!]_{\mathbb{X}}$ is contained in the basic open

$$[\![x = m]\!]_{\mathbb{X}} = \{ \langle m, M' \rangle \mid m \in M' \},$$

where m is a parameter indexing n. The image of $[\![x = m]\!]_{\mathbb{X}}$ under the map

$$\pi_{[\![x:\top]\!]} \colon [\![x : \top]\!]_{\mathbb{X}} \longrightarrow X_0^{\tau\text{-}\log_0}$$

is open, namely $\pi_{[\![x:\top]\!]}([\![x = m]\!]_{\mathbb{X}}) = [\![m : \top]\!]_{\mathbb{X}}$. Moreover, there is an evident local section $s \colon [\![m : \top]\!]_{\mathbb{X}} \to [\![x = m]\!]_{\mathbb{X}}$ of $\pi_{[\![x:\top]\!]}$ – the map that sends $M \in X_0$, in which the parameter m is realised, to the pair $\langle n, M \rangle$, where n realises the parameter m. It remains to show that the local section s is continuous, but this is clear since

$$\begin{aligned} s^{-1}\left([\![x = m]\!]_{\mathbb{X}} \cap [\![x, \vec{m}' : \psi]\!]_{\mathbb{X}}\right) &= \left\{ M \in X_0 \,\middle|\, M \vDash \psi(m, \vec{m}') \right\}, \\ &= [\![m, \vec{m}' : \psi]\!]_{\mathbb{X}} \subseteq [\![m : \top]\!]_{\mathbb{X}}. \end{aligned}$$

Hence, for the topology T_x on $[\![x : \top]\!]_{\mathbb{X}}$, the map $\pi_{[\![x:\top]\!]} \colon [\![x : \top]\!]_{\mathbb{X}} \to X_0^{\tau\text{-}\log_0}$ is a local homeomorphism.

We claim that the product topology $T_{\vec{x}}$ on $[\![\vec{x} : \top]\!]_{\mathbb{X}}$ is the topology generated by the basis whose opens are definables with parameters. Clearly, if the product topology $T_{\vec{x}}$ is, as claimed, generated by definables with parameters, then each definable *without* parameters $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$ is an open subset as desired.

The inclusion of the product topology $T_{\vec{x}}$ in the topology generated by definables with parameters is obtained by noting that

$$\begin{aligned} \prod_{x_i \in \vec{x}} [\![x_i, \vec{m}_i : \varphi_i]\!]_{\mathbb{X}} &= \{ \langle \vec{n}, M \rangle \,|\, \forall i \, M \vDash \varphi_i(n_i, \vec{m}_i) \}, \\ &= \left\{ \langle \vec{n}, M \rangle \,\middle|\, \vec{n}, \vec{m}_1, \dots \vec{m}_n \in \left[\!\!\left[\vec{y}, \vec{x}_1, \dots, \vec{x}_n : \bigwedge_{i=1}^{n} \varphi_i \right]\!\!\right]_M \right\}, \\ &= \left[\!\!\left[\vec{x}, \vec{m}_1, \dots, \vec{m}_n : \bigwedge_{i=1}^{n} \varphi_i \right]\!\!\right]_{\mathbb{X}}. \end{aligned}$$

For the reverse inclusion, we observe that, for each $\langle \vec{n}, M \rangle \in [\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}}$,

$$\begin{aligned} \langle \vec{n}, M \rangle \in \prod_{x_i \in \vec{x}} & [\![x_i : x_i = m_i' \wedge \varphi(\vec{m}', \vec{m})]\!]_{\mathbb{X}}, \\ &= [\![\vec{x} : \vec{x} = \vec{m}' \wedge \varphi(\vec{m}', \vec{m})]\!]_{\mathbb{X}} \subseteq [\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}}, \end{aligned}$$

where $\vec{m}'$ is a tuple of parameters indexing $\vec{n} \in M$. □

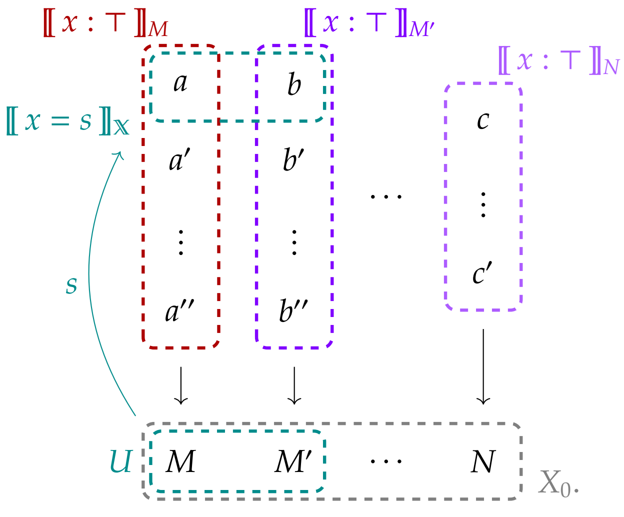
Every factoring topology for objects contains the logical topology. As expressed below, there is a sense in which the 'logical topology for objects' is essentially the only factoring topology that need be considered.

Proposition VII.17. *Let $\mathbb{T}$ be a geometric theory and let X_0 be a set of models of $\mathbb{T}$. A topology τ_0 on X_0 is a factoring topology for objects if and only if there exists an indexing of each $M \in X_0$ by a set of parameters $\mathfrak{K}$ such that τ_0 contains the corresponding 'logical topology for objects' $\tau\text{-log}_0$.*

Proof. One direction is clear: if τ_0 contains a 'logical topology for objects' $\tau\text{-log}_0$ then, by Lemma VII.16, the geometric morphism $\mathbf{p}_0$ factors as

$$\mathbf{Sh}(X_0^\delta) \longrightarrow \mathbf{Sh}(X_0^{\tau_0}) \longrightarrow \mathbf{Sh}\left(X_0^{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}.$$

When we assume instead that τ_0 is a factoring topology for objects, then, in particular, for each singleton context, the projection $\pi_{[\![x:\top]\!]} \colon [\![x : \top]\!]_{\mathbb{X}} \to X_0^{\tau_0}$ is a local homeomorphism for some choice of topology T_x on $[\![x : \top]\!]_{\mathbb{X}}$. In particular, every element $\langle n, M \rangle$ of $[\![x : \top]\!]_{\mathbb{X}}$ lies in the image of some local section $s \colon U \to [\![x : \top]\!]_{\mathbb{X}}$ of $\pi_{[\![x:\top]\!]}$ whose domain and image are open We call such local sections *open*. Let $\mathfrak{K}$ be the set of parameters whose elements are open local sections of $\pi_{[\![x:\top]\!]}$. We can index each $M \in X_0$ by interpreting the parameter $s \colon U \to [\![x : \top]\!]_{\mathbb{X}}$ by $n \in M$ if $\langle n, M \rangle$ lies in the image $s(U)$. Thus, the open $s(U) \subseteq [\![x : \top]\!]_{\mathbb{X}}$ is the interpretation of the definable with parameters $[\![x = s]\!]_{\mathbb{X}}$, as in the diagram



As the open local sections are jointly surjective, this does indeed define an indexing of each $M \in X_0$ by the parameters $\mathfrak{K}$.

It remains to show that τ_0 contains the logical topology for this indexing. We note that $s(U) = [\![x = s]\!]_{\mathbb{X}}$ is open in $[\![x : \top]\!]_{\mathbb{X}}$ and therefore, by Proposition VII.12 (and its reformulation in Remark VII.13), in the product topology on $[\![\vec{x} : \top]\!]_{\mathbb{X}}$, for any tuple of parameters $\vec{s}$ of the same sort as $\vec{x}$ and any formula in context $\{ \vec{x} : \varphi \}$, the subset

$$\begin{aligned}[\![\vec{x} : \varphi \wedge \vec{x} = \vec{s}]\!]_{\mathbb{X}} &= [\![\vec{x} : \varphi]\!]_{\mathbb{X}} \cap \prod_{x_i \in \vec{x}} [\![x_i = s_i]\!], \\ &= \{ \langle \vec{n}, M \rangle \mid M \vDash \varphi(\vec{s}),\ \vec{n} = \vec{s} \}\end{aligned}$$

is open in the product topology on $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$. Since the local homeomorphism

$$\pi_{[\![\, x:\top \,]\!]} \colon [\![\, x : \top \,]\!]_{\mathbb{X}} \longrightarrow X_0^{\tau\text{-}\log_0}$$

is, in particular, an open map, the image

$$\pi_{[\![\, \vec{x}:\top \,]\!]}([\![\, \vec{x} : \varphi \wedge \vec{x} = \vec{s} \,]\!]_{\mathbb{X}}) = [\![\, \vec{s} : \varphi[\vec{s}/\vec{x}] \,]\!]_{\mathbb{X}}$$

is open in τ_0, i.e. τ_0 contains the 'logical topology for objects' as desired. $\square$

Remark VII.18. Let $\mathbb{T}$ be geometric theory over a signature Σ, let X_0 be a set of $\mathbb{T}$-models, and let τ_0 be a factoring topology for objects on X_0.

(i) When we discover an indexing of X_0 by the set of parameters $\mathfrak{K}$ such that τ_0 contains the 'logical topology for objects' τ-$\log_0$ via the method of Proposition VII.17, we note that we have also forced the chosen topology on $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$ to contain as opens the definables with parameters $[\![\, \vec{x}, \vec{m} : \varphi \,]\!]_{\mathbb{X}}$.

(ii) If $\mathbb{T}$ is an *inhabited theory*, meaning that $\mathbb{T}$ proves the sequent $\top \vdash_{\emptyset} \exists x \top$ (for each sort of Σ), then each fibre of $[\![\, x : \top \,]\!]_{\mathbb{X}}$ is non-empty. Thus, every open in τ_0 is of the form $\pi_{[\![\, x:\top \,]\!]}(V)$ for some open $V \subseteq [\![\, x : \top \,]\!]_{\mathbb{X}}$. In particular, the sets of the form
$$U = \pi_{[\![\, x:\top \,]\!]}([\![\, x = s \,]\!]_{\mathbb{X}}) = [\![\, s : \top \,]\!]_{\mathbb{X}},$$
for $s \colon U \to [\![\, x : \top \,]\!]_{\mathbb{X}}$ an open local section of $\pi_{[\![\, x:\top \,]\!]}$, form a basis for τ_0. Therefore, when each $M \in \mathbb{X}_0$ is indexed by the open local sections of $\pi_{[\![\, x:\top \,]\!]}$ as in Proposition VII.17, the induced logical topology τ-$\log_0$ contains τ_0 too, and so $\tau_0 = \tau$-$\log_0$.

VII.3 Factoring topologies for arrows

Let $\mathbb{T}$ be a geometric theory, $\mathbb{X} = (X_1 \rightrightarrows X_0)$ a model groupoid for $\mathbb{T}$, and let τ_0 be a factoring topology for objects on X_0. We seek to classify the possible factoring topologies for arrows on X_1. As before, we first note that there is a factorisation

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) \arrow[d, "w"'] \arrow[dr, "\mathbf{p}'"] & \\ \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \arrow[r, dashed] & \mathcal{E}_{\mathbb{T}} \end{tikzcd}$$

if and only if there is a factorisation

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{X}^{\delta}_{\tau_0}) & \\ \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \arrow[u, "W"] & \mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}, \arrow[l, dashed] \arrow[ul, "\mathbf{p}'^{*} \circ \ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}"'] \end{tikzcd}$$

if and only if, for each object $(\vec{x}, \varphi)$ of $\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}$, the X_1^{δ}-action on $[\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}$ is continuous. In fact, it suffices to only check that, for each context $\vec{x}$, the action

$$\theta_{[\![\, \vec{x}:\top \,]\!]} \colon [\![\, \vec{x} : \top \,]\!]_{\mathbb{X}} \times_{X_0} X_1^{\tau_1} \longrightarrow [\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$$

is continuous since then, for each formula in context $\{\vec{x} : \varphi\}$, the restriction of $\theta_{[\![\vec{x}:\top]\!]}$ to the subspace $[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \subseteq [\![\vec{x} : \top]\!]_{\mathbb{X}}$, i.e. the X_1-action

$$\theta_{[\![\vec{x}:\varphi]\!]} \colon [\![\vec{x} : \varphi]\!]_{\mathbb{X}} \times_{X_0} X_1^{\tau_1} \longrightarrow [\![\vec{x} : \varphi]\!]_{\mathbb{X}},$$

is continuous as well.

Logical topology for arrows. We define the logical topology for arrows, another variation on a topology utilised in [5], [17], [36], [37]. Much like the logical topology for objects, we will observe in Proposition VII.22 that the logical topology for arrows plays a special role among all factoring topologies for arrows.

Definition VII.19 (Definition 3.1 [37], Definition 1.2.1 [5]). Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for a geometric theory $\mathbb{T}$ indexed by the set of parameters $\mathfrak{K}$. The *logical topology for arrows* is the topology on X_1 generated by basic opens of the form

$$\left[\!\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{array}\right]\!\!\right]_{\mathbb{X}} = \left\{ M \xrightarrow{\alpha} N \in X_1 \;\middle|\; \begin{array}{c} M \vDash \varphi(\vec{a}), \\ N \vDash \alpha(\vec{b}) = \vec{c}, \\ N \vDash \psi(\vec{d}) \end{array} \right\},$$

where $\{\vec{x} : \varphi\}, \{\vec{y} : \psi\}$ are formulae, and $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are tuples of parameters in $\mathfrak{K}$.

Lemma VII.20. *The groupoid* $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$,

$$\begin{tikzcd} X_1^{\tau\text{-log}_1} \times_{X_0} X_1^{\tau\text{-log}_1} \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m"] \arrow[r, shift right=2, "\pi_1"'] & X_1^{\tau\text{-log}_1} \arrow[loop below, "i"] \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & X_0^{\tau\text{-log}_0} \arrow[l, "e"'] \end{tikzcd},$$

is a topological groupoid, i.e. the maps s, t, etc., are continuous.

Proof. For each basic open subset of $X_0^{\tau\text{-log}_0}$ and $X_1^{\tau\text{-log}_1}$, we have that

$$s^{-1}([\![\vec{m} : \varphi]\!]_{\mathbb{X}}) = \left[\!\!\left[\begin{array}{c} \vec{m} : \varphi \\ \emptyset \mapsto \emptyset \\ \emptyset : \top \end{array}\right]\!\!\right]_{\mathbb{X}},$$

$$e^{-1}\left(\left[\!\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{array}\right]\!\!\right]_{\mathbb{X}}\right) = [\![\vec{a} : \varphi]\!]_{\mathbb{X}} \cap [\![\vec{d} : \psi]\!]_{\mathbb{X}} \cap [\![\vec{b} = \vec{c}]\!]_{\mathbb{X}},$$

$$\text{and } m^{-1}\left(\left[\!\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{array}\right]\!\!\right]_{\mathbb{X}}\right) = \bigcup_{\vec{e} \in \mathfrak{K}} \left[\!\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{e} \\ \emptyset : \top \end{array}\right]\!\!\right]_{\mathbb{X}} \times_{X_0} \left[\!\!\left[\begin{array}{c} \emptyset : \top \\ \vec{e} \mapsto \vec{c} \\ \vec{d} : \psi \end{array}\right]\!\!\right]_{\mathbb{X}}.$$

The continuity of the other arrows is just as easily checked. □

We are now able to recognise the logical topology for arrows as a factoring topology for arrows. Below, we explore the special role the logical topology plays among all factoring topologies.

Lemma VII.21 (Lemma 2.3.4.3 [36])**.** *When X_0 is endowed with the logical topology on objects $\tau\text{-log}_0$, then the logical topology on arrows $\tau\text{-log}_1$ is a factoring topology for arrows.*

Proof. By the above discussion, it suffices to check that the X_1-action $\theta_{[\![\vec{x}:\top]\!]}$ is continuous for each context $\vec{x}$, i.e. we must show that, for each definable with parameters $[\![\vec{x},\vec{m}:\varphi]\!]_{\mathbb{X}}$,

$$\theta^{-1}_{[\![\vec{x}:\top]\!]}([\![\vec{x},\vec{m}:\varphi]\!]_{\mathbb{X}}) = \left\{ (\langle \vec{n}, M\rangle, \alpha) \,\middle|\, M \xrightarrow{\alpha} N \in X_1,\ N \vDash \varphi(\alpha(\vec{n}),\vec{m}) \right\}$$

is an open subset of $[\![\vec{x}:\top]\!]_{\mathbb{X}} \times_{X_0} X_1^{\tau\text{-log}_0}$. This is easily demonstrated. For each $(\langle \vec{n}, M\rangle, \alpha) \in \theta^{-1}_{[\![\vec{x}:\top]\!]}([\![\vec{x},\vec{m}:\varphi]\!]_{\mathbb{X}})$, we note that

$$(\langle \vec{n}, M\rangle, \alpha) \in [\![\vec{x},\vec{m}':\varphi]\!]_{\mathbb{X}} \times_{X_0} \left[\!\!\left[\begin{array}{c} \emptyset:\top \\ \vec{m}' \mapsto \vec{m} \\ \emptyset:\top \end{array}\right]\!\!\right]_{\mathbb{X}} \subseteq \theta^{-1}_{[\![\vec{x}:\top]\!]}([\![\vec{x},\vec{m}:\varphi]\!]_{\mathbb{X}}),$$

where $\vec{m}'$ is a tuple of parameters indexing $\alpha^{-1}(\vec{m})$. □

Proposition VII.22. *Let $\mathbb{T}$ be a geometric theory, $\mathbb{X} = (X_1 \rightrightarrows X_0)$ a model groupoid for $\mathbb{T}$, and let τ_0 be a factoring topology for objects and τ_1 a factoring topology for arrows. If τ_0 contains the logical topology for objects when $\mathbb{X}$ is indexed by a set of parameters $\mathfrak{K}$, then τ_1 contains the logical topology on arrows.*

Proof. We note that

$$\left[\!\!\left[\begin{array}{c} \vec{a}:\varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d}:\psi \end{array}\right]\!\!\right]_{\mathbb{X}} = s^{-1}([\![\vec{a}:\varphi]\!]_{\mathbb{X}}) \cap \left[\!\!\left[\begin{array}{c} \emptyset:\top \\ \vec{b} \mapsto \vec{c} \\ \emptyset:\top \end{array}\right]\!\!\right]_{\mathbb{X}} \cap t^{-1}([\![\vec{d}:\psi]\!]_{\mathbb{X}}).$$

As s, t are continuous maps and $[\![\vec{a}:\varphi]\!]_{\mathbb{X}}, [\![\vec{d}:\psi]\!]_{\mathbb{X}}$ are open in $X_0^{\tau_0}$, we deduce that

$$s^{-1}([\![\vec{a}:\varphi]\!]_{\mathbb{X}}), t^{-1}([\![\vec{d}:\psi]\!]_{\mathbb{X}}) \subseteq X_1^{\tau_1}$$

are open. Thus, it suffices to show that τ_1 contains the subset

$$\left[\!\!\left[\begin{array}{c} \emptyset:\top \\ \vec{b} \mapsto \vec{c} \\ \emptyset:\top \end{array}\right]\!\!\right]_{\mathbb{X}} = \left\{ M \xrightarrow{\alpha} N \in X_1 \,\middle|\, M \vDash \alpha(\vec{b}) = \vec{c} \right\}.$$

By Remark VII.18, we may assume that the topology $T_{\vec{x}}$ on $[\![\vec{x}:\top]\!]_{\mathbb{X}}$, for which $\pi_{[\![\vec{x}:\top]\!]} : [\![\vec{x}:\top]\!]_{\mathbb{X}}^{T_{\vec{x}}} \to X_0^{\tau_0}$ is a local homeomorphism, contains as opens the definables with parameters $[\![\vec{x},\vec{m}:\varphi]\!]_{\mathbb{X}}$.

Since $\theta_{[\![\vec{x}:\top]\!]}$ is continuous, the subset

$$\theta^{-1}_{[\![\vec{x}:\top]\!]}([\![x = \vec{c}]\!]_{\mathbb{X}}) \cap \left([\![\vec{x} = \vec{b}]\!]_{\mathbb{X}} \times_{X_0} X_1\right) = \{ (\langle \vec{n}, M\rangle, \alpha) \mid \vec{n} = \vec{b},\ \alpha(\vec{n}) = \vec{c} \}$$

is open in $[\![\vec{x}:\top]\!]_{\mathbb{X}} \times_{X_0} X_1^{\tau_1}$. The projection pr_2 in the pullback square

$$\begin{tikzcd} {[\![\vec{x}:\top]\!]_{\mathbb{X}} \times_{X_0} X_1^{\tau_1}} \arrow[r] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\vec{x}:\top]\!]_{\mathbb{X}}} \arrow[d, "\pi_{[\![\vec{x}:\top]\!]}"] \\ X_1^{\tau_1} \arrow[r, "s"] & X_0^{\tau_0} \end{tikzcd}$$

is an open map since the local homeomorphism $\pi_{[\![\vec{x} : \top]\!]} \colon [\![\vec{x} : \top]\!]_X \to X_0^{\tau_0}$ is an open map too, and open maps are stable under pullback. Therefore,

$$\mathrm{pr}_2\left(\theta^{-1}_{[\![\vec{x}:\top]\!]}([\![x = \vec{c}]\!]_{\mathbb{X}}) \cap \left([\![\vec{x} = \vec{b}]\!]_{\mathbb{X}} \times_{X_0} X_1\right)\right) = \left[\!\!\left[\begin{matrix} \emptyset : \top \\ \vec{b} \mapsto \vec{c} \\ \emptyset : \top \end{matrix} \right]\!\!\right]_{\mathbb{X}}$$

is an open subset of $X_1^{\tau_1}$, and thus τ_1 contains the logical topology on arrows. □

VII.3.1 Characterising the logical topology for arrows

Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ a model groupoid for $\mathbb{T}$ indexed by $\mathfrak{K}$. By Lemma VII.21 and Proposition VII.22, for any open factoring topology on arrows τ_1, when X_0 is endowed with the logical topology for objects $\tau\text{-}\log_0$, there is a factorisation of the geometric morphism

$$\mathbf{p}' \colon \mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-}\log_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

as

$$\mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}.$$

Moreover, by Proposition V.14 and Remark V.15, the factoring geometric morphisms

$$\mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau\text{-}\log_0}\right) \quad \text{and} \quad \mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right)$$

are both hyperconnected morphisms. We may therefore wonder whether the the factorisation

$$\mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}. \tag{VII.ii}$$

is the hyperconnected-localic factorisation of the geometric morphism $\mathbf{p}'$.

We answer affirmatively under the condition that

$$\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0} = \left(X_1^{\tau\text{-}\log_1} \rightrightarrows X_0^{\tau\text{-}\log_0}\right)$$

is an *open* topological groupoid. In general, there is no reason for $\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}$ to be an open topological groupoid, though the groupoid eliminating imaginaries is a sufficient condition, as observed in Lemma VII.28.

The proof that (VII.ii) is the hyperconnected-localic factorisation is essentially contained in Lemmas 2.3.4.10-13 of [36], which deal with the specific case of Forssell groupoids (which we will study in more detail in Section VII.5.4). We sketch some of the details of the proof to assure the reader that the only required assumption is that $\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}$ is an open topological groupoid.

Lemma VII.23 (Lemma 2.3.4.10 [36])**.** *Let (Y, β, q) be an $\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}$-sheaf. For each $y \in Y$, there exists a basic open $[\![\vec{m} : \xi]\!]_{\mathbb{X}}$ of $X_0^{\tau\text{-}\log_0}$ with an open local section $f \colon [\![\vec{m} : \xi]\!]_{\mathbb{X}} \to Y$ of q such that*

(i) *the point y is in the image of f,*

(ii) *for any $M \in \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}}$ and an isomorphism $M \xrightarrow{\alpha} N \in X_1$, if $\alpha(\vec{m}) = \vec{m}$ (and so $N \in \llbracket \vec{m} : \xi \rrbracket$), then*
$$\beta\,(f(M), \alpha) = f(N).$$

Recall that the subobjects of a $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$-sheaf (Y, q, β) are given precisely by the open subsets of Y that are stable under the action of β. If $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is an open topological groupoid then, by Lemma V.4, the $X_1^{\tau\text{-log}_1}$-action $\theta_{\llbracket \vec{x} : \top \rrbracket}$ is an open map. Therefore, we have that
$$\overline{\llbracket \vec{x}, \vec{m} : \varphi \rrbracket_{\mathbb{X}}} = \theta_{\llbracket \vec{x} : \top \rrbracket}(\llbracket \vec{x}, \vec{m} : \varphi \rrbracket_{\mathbb{X}} \times_{X_0} X_1) \subseteq \llbracket \vec{x} : \top \rrbracket_{\mathbb{X}},$$
the orbit of a definable with parameters $\llbracket \vec{x}, \vec{m} : \varphi \rrbracket_{\mathbb{X}}$, is an open, stable subset and hence a subobject of $\llbracket \vec{x} : \top \rrbracket$.

Proposition VII.24 (Lemmas 2.3.4.11-13 [36])**.** *If*
$$\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0} = \left(X_1^{\tau\text{-log}_1} \rightrightarrows X_0^{\tau\text{-log}_0}\right)$$
is an open topological groupoid, the factoring geometric morphism
$$\mathbf{p}^{\log} \colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$
is localic.

We sketch the proof to Proposition VII.24. To show that $\mathbf{p}$ is localic, it suffices to show that the subobjects of $\llbracket \vec{x} : \top \rrbracket_{\mathbb{X}}$ form a generating set of $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$. Given an object (Y, q, β) of $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$ and a point y of Y, by Lemma VII.23, there exists a basic open $\llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}}$ of $X_0^{\tau\text{-log}_0}$ with a local section $f \colon \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}} \to Y$ of q such that

(i) the point y is in the image of f,

(ii) for any $M \in \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}}$ and an isomorphism $M \xrightarrow{\alpha} N \in X_1$, if $\alpha(\vec{m}) = \vec{m}$ then
$$\beta\,(f(M), \alpha) = f(N).$$

Let $\vec{x}$ be a context with the same type as $\vec{m}$. Evidently, there is a local section $g \colon \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}} \to \llbracket \vec{x} : \top \rrbracket_{\mathbb{X}}$ of $\pi_{\llbracket \vec{x} : \top \rrbracket} \colon \llbracket \vec{x} : \top \rrbracket_{\mathbb{X}} \to X_0$ that sends $M \in \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}}$ to $\langle \vec{m}, M \rangle$. The image of g is thus the open subset $\llbracket \vec{x} : \vec{x} = \vec{m} \wedge \xi \rrbracket_{\mathbb{X}} \subseteq \llbracket \vec{x} : \top \rrbracket_{\mathbb{X}}$. Hence, there is a commuting diagram of continuous maps

$$\begin{tikzcd}
\llbracket \vec{x} : \top \rrbracket_{\mathbb{X}} & \overline{\llbracket \vec{x} : \vec{x} = \vec{m} \wedge \xi \rrbracket_{\mathbb{X}}} \arrow[l, tail] & Y \\
& \llbracket \vec{x} : \vec{x} = \vec{m} \wedge \xi \rrbracket_{\mathbb{X}} \arrow[u, tail] \arrow[ul, tail, bend left] & \\
& \llbracket \vec{m} : \xi \rrbracket_{\mathbb{X}}. \arrow[u] \arrow[uul, bend left, "g"] \arrow[uur, bend right, "f"'] &
\end{tikzcd}$$

The remainder of the proof of Proposition VII.24 consists of constructing a continuous map $h\colon \overline{[\![\, \vec{x} = \vec{m} \wedge \xi \,]\!]}_{\mathbb{X}} \to Y$ which completes the above diagram and moreover constitutes a morphism of $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$-sheaves.

As each element of $\overline{[\![\, \vec{x} = \vec{m} \wedge \xi \,]\!]}_{\mathbb{X}}$ is of the form $\langle \alpha(\vec{m}), N \rangle$ where $M \xrightarrow{\alpha} N$ is a $\mathbb{T}$-model isomorphism in X_1 and $M \in [\![\, \vec{m} : \xi \,]\!]$, we take the obvious definition and set $h(\langle \alpha(\vec{m}), N \rangle) = \beta(f(M), \alpha)$.

It must first be checked that this is well-defined, and here we use the fact that $\mathbb{X}$ is a groupoid. Given a second isomorphism $M \xrightarrow{\gamma} N$ such that $\langle \alpha(\vec{m}), N \rangle = \langle \gamma(\vec{m}), N \rangle$, then $\gamma^{-1} \circ \alpha$ is an automorphism of M, contained in X_1, such that $\gamma^{-1} \circ \alpha(\vec{m}) = \vec{m}$. Hence, by hypothesis, $\beta(f(M), \gamma^{-1} \circ \alpha) = f(M)$, and so

$$\beta(f(M), \gamma) = \beta(\beta(f(M), \gamma^{-1} \circ \alpha), \gamma) = \beta(f(M), \alpha).$$

It remains to show that h is continuous and that h is a morphism of $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$-sheaves. For these details, the reader is directed to [36].

Thus, for each object (Y, q, β) of $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$, the arrows of $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$ whose domains are subobjects of $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$ are jointly surjective and therefore the geometric morphism

$$\mathbf{p}^{\log}\colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is localic.

Corollary VII.25. *If $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is an open topological groupoid, the geometric morphism*

$$\mathbf{p}^{\log}\colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is the localic part of the hyperconnected-localic factorisation of

$$\mathbf{p}'\colon \mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

Proof. There is a commutative triangle

$$\begin{tikzcd} \mathbf{Sh}\left(\mathbb{X}^{\delta}_{\tau\text{-log}_0}\right) \arrow[r, "w"] \arrow[dr, "\mathbf{p}'"'] & \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \arrow[d, "\mathbf{p}^{\log}"] \\ & \mathcal{E}_{\mathbb{T}}, \end{tikzcd}$$

where w is hyperconnected by Proposition V.14 and $\mathbf{p}^{\log}$ is localic by Proposition VII.24. □

VII.4 The proof of the classification theorem

We are now in a position to combine the results of Section VII.2 and Section VII.3 to obtain the classification theorem stated in Theorem VII.8. We separate the different steps of the proof to show clearly the interaction between the two conditions: conservativity and elimination of parameters. Conservativity, unsurprisingly, is equivalent to the induced geometric morphism being a geometric surjection. Conversely, elimination of parameters is equivalent the induced geometric morphism being a geometric embedding. Finally, we also give a sense in which the logical topologies are the only topologies that need be considered.

Lemma VII.26. *Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{X}$. Given a pair of factoring topologies τ_0 on X_0 and τ_1 on X_1, the factoring geometric morphism*

$$\mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is a geometric surjection if and only if X_0 is a conservative set of models for $\mathbb{T}$.

Proof. Recall that there is commutative diagram of geometric morphisms

$$\begin{tikzcd}
\mathbf{Sets}^{X_0} \simeq \mathbf{Sh}(X_0^{\delta}) \arrow[r, "j"] \arrow[rrd, "\mathbf{p}_0"'] & \mathbf{Sh}(X_0^{\tau_0}) \arrow[r, "u"] & \mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau_0}\right) \arrow[d] \\
 & & \mathcal{E}_{\mathbb{T}}
\end{tikzcd}$$

where the top horizontal composite $u \circ j$ is a geometric surjection. Recall also that X_0 is a conservative set of models for $\mathbb{T}$ if and only if the geometric morphism $\mathbf{p}_0$ is a surjection.

Thus, using the fact that geometric surjections are the left class in an orthogonal factorisation system – namely, the (surjection, inclusion)-factorisation (see [63, §A4.2]), and are therefore closed under composites and have the right cancellation property (if $f \circ g$ and g are surjections, then so is f), we conclude that $\mathbf{Sh}\left(\mathbb{X}^{\tau_1}_{\tau_0}\right) \to \mathcal{E}_{\mathbb{T}}$ is a geometric surjection if and only if X_0 is a conservative set of models for $\mathbb{T}$. ◻

From a representing groupoid to elimination of parameters. We now continue with the proof for one implication of Theorem VII.8. We first show that an open representing model groupoid can be given an indexing by parameters for which the groupoid is conservative and eliminates parameters.

Proposition VII.27. *Let $\mathbb{T}$ be a geometric theory, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$. If $\mathbb{X}$ is an open representing groupoid, i.e. there exist topologies on X_0 and X_1 making $\mathbb{X}$ an open topological groupoid such that $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$, then there exist a set of parameters $\mathfrak{K}$ and an indexing of $\mathbb{X}$ by $\mathfrak{K}$ for which $\mathbb{X}$ is conservative and eliminates parameters.*

Proof. We apply Proposition VII.17 and Remark VII.18 to deduce that there exists an indexing of $\mathbb{X}$ by parameters $\mathfrak{K}$ and that we can assure that the space $[\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}$, i.e. the image of $(\vec{x}, \top)$ under the functor

$$\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}} \xrightarrow{\ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}} \mathbf{Sh}(\mathbf{Con}_N \rtimes F^{\mathbb{T}}, K_{F^{\mathbb{T}}}) \simeq \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{X}),$$

contains as open subsets the definables with parameters.

Since $\mathbb{X}$ is an open groupoid, the orbit of each open subset of the form $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ is still an open subset of $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$. Therefore, being a stable open, $\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}}$ defines a subobject of $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$. Under the equivalence $\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{X})$, the subobjects of $\ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\, \vec{x}\,,\ \top\,)$ and $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$ must also be identified. Hence, recalling from Section II.2 (cf. [63, Lemma D1.4.4(iv)] as well) that the subobjects of $\ell_{\mathbf{Con}_N^{\mathrm{op}} \rtimes F^{\mathbb{T}}}(\, \vec{x}\,,\ \top\,)$ are formulae in the context $\vec{x}$, we deduce that there exists a formula φ such that

$$\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}} = [\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}.$$

Therefore, the groupoid $\mathbb{X}$, as indexed by $\mathfrak{K}$, eliminates parameters.

Finally, since an equivalence of topoi is, in particular, a surjection, we can apply Lemma VII.26 to deduce that $\mathbb{X}$ is also conservative. $\square$

From elimination of parameters to a representing groupoid. We now prove the converse statement of Theorem VII.8 that an indexed model groupoid that is conservative and eliminates parameters yields an open representing groupoid. Unsurprisingly, the topologies we consider are the logical topologies studied in Sections VII.2 to VII.3. We first demonstrate that the condition that the model groupoid eliminates parameters is equivalent to the induced geometric morphism being an inclusion of a subtopos.

Lemma VII.28. *Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$ indexed by parameters $\mathfrak{K}$. If $\mathbb{X}$ eliminates parameters, then, when both X_1 and X_0 are endowed with the logical topologies, $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ becomes an* open *topological groupoid.*

Proof. We have already seen that $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is a topological groupoid in Lemma VII.20, so it remains to show that either of the continuous maps $s, t \colon X_1^{\tau\text{-log}_1} \rightrightarrows X_0^{\tau\text{-log}_0}$ are open too. We will show that t is open.

It suffices to show that the image of each basic open of $X_1^{\tau\text{-log}_1}$ is open in $X_0^{\tau\text{-log}_0}$. Suppose that

$$N \in t\left(\left[\!\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{array} \right]\!\!\right]_{\mathbb{X}} \right).$$

Then there is some isomorphism $M \xrightarrow{\alpha} N$ of X_1 such that $M \vDash \varphi(\vec{a})$ and $\alpha(\vec{b}) = \vec{c}$, in addition to $N \vDash \psi(\vec{d})$. Therefore,

$$\langle \vec{c}, N \rangle \in \overline{\left[\!\!\left[\, \vec{x}, \vec{a} : \vec{b} = \vec{x} \wedge \varphi \,\right]\!\!\right]_{\mathbb{X}}}.$$

Since $\mathbb{X}$ eliminates parameters, there is some formula χ without parameters such that

$$\overline{\left[\!\!\left[\, \vec{x}, \vec{a} : \vec{b} = \vec{x} \wedge \varphi \,\right]\!\!\right]_{\mathbb{X}}} = [\![\, \vec{x} : \chi \,]\!]_{\mathbb{X}}.$$

We thus conclude that N is contained in the open subset $\left[\!\!\left[\, \vec{c}, \vec{d} : \chi \wedge \psi \,\right]\!\!\right]_{\mathbb{X}}$ of $X_0^{\tau\text{-log}_0}$.

Given any other $N' \in [\![\, \vec{c}, \vec{d} : \chi \wedge \psi \,]\!]_{\mathbb{X}}$, we have that

$$\langle \vec{c}, N' \rangle \in [\![\, \vec{x} : \chi \,]\!]_{\mathbb{X}} = \overline{\left[\!\!\left[\, \vec{x}, \vec{a} : \vec{b} = \vec{x} \wedge \varphi \,\right]\!\!\right]_{\mathbb{X}}}.$$

Thus, there exists another isomorphism $M' \xrightarrow{\gamma} N'$ of X_1 such that $M' \vDash \varphi(\vec{a})$ and $\gamma(\vec{b}) = \vec{c}$. Hence, we have that

$$N, N' \in \left[\!\left[\, \vec{c}, \vec{d} : \chi \wedge \psi \,\right]\!\right]_{\mathbb{X}} \subseteq t\left(\left[\!\left[\begin{array}{c} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{array} \right]\!\right]_{\mathbb{X}}\right),$$

and thus the topological groupoid $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is open. $\square$

Proposition VII.29. *Let $\mathbb{T}$ be a geometric theory, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$ indexed by parameters $\mathfrak{K}$. The factoring geometric morphism $\mathbf{p}^{\log}$ is an inclusion of a subtopos*

$$\mathbf{p}^{\log} \colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \rightarrowtail \mathcal{E}_{\mathbb{T}}$$

if and only if $\mathbb{X}$ eliminates parameters.

Proof. The first step is to deduce that, under either hypothesis, the geometric morphism $\mathbf{p}^{\log}$ is a localic geometric morphism. This is clear if $\mathbf{p}^{\log}$ is an inclusion of a subtopos since every inclusion is localic (see [63, Examples A4.6.2(a)]). Conversely, if $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ eliminates parameters then, by Lemma VII.28, $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is an open topological groupoid. Thus, by applying Proposition VII.24, the factoring geometric morphism

$$\mathbf{p}^{\log} \colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is a localic geometric morphism.

Recall that $\mathcal{E}_{\mathbb{T}}$ is the topos of internal sheaves on the internal locale $F^{\mathbb{T}}$ of $\mathbf{Sets}^{\mathbf{Con}_N}$, and thus there is localic geometric morphism $C_{\pi_{F^{\mathbb{T}}}} \colon \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(F^{\mathbb{T}}) \to \mathbf{Sets}^{\mathbf{Con}_N}$. Since localic geometric morphisms are closed under composition (see [59, Lemma 1.1]), $\mathbf{Sh}(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0})$ is also localic over $\mathbf{Sets}^{\mathbf{Con}_N}$. Indeed, by Proposition VII.24, it is the topos of sheaves on the internal locale

$$\mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)}([\![\, - : \top \,]\!]_{\mathbb{X}}) \colon \mathbf{Con}_N \longrightarrow \mathbf{Frm}_{\mathrm{open}}.$$

Hence, the localic geometric morphism $\mathbf{p}^{\log}$ in Section VII.4 is induced by a morphism of internal locales by Proposition II.23, namely the internal locale morphism whose component at a context $\vec{x}$ is given by the frame homomorphism

$$\begin{aligned} [\![\, - \,]\!]_{\mathbb{X}\vec{x}} \colon F^{\mathbb{T}}(\vec{x}) &\to \mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)}([\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}), \\ \varphi &\mapsto [\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}}. \end{aligned}$$

By Theorem II.34, the geometric morphism $\mathbf{p}^{\log}$ induced by the internal locale morphism $[\![\, - \,]\!]_{\mathbb{X}}$ is an inclusion if and only if $[\![\, - \,]\!]_{\mathbb{X}}$ is an internal sublocale embedding, i.e. $[\![\, - \,]\!]_{\mathbb{X}\vec{x}}$ is surjective for each context $\vec{x}$.

Recall that a subobject of $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$ is an open subset U that is stable under the action $\theta_{[\![\, \vec{x}:\top \,]\!]}$. As the opens of the form $[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}$ form a basis for the topology on $[\![\, \vec{x} : \top \,]\!]_{\mathbb{X}}$, every stable open subset U is of the form

$$U = \bigcup_{i \in I} \overline{[\![\, \vec{x}, \vec{m}_i : \psi_i \,]\!]_{\mathbb{X}}}.$$

Therefore, the stable opens of the form $\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}}$ form a basis for the frame of subobjects

$$\mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)}([\![\, \vec{x} : \top \,]\!]_{\mathbb{X}})$$

and so $[\![\, - \,]\!]_{\mathbb{X}_{\vec{x}}}$ is surjective if and only if every basic subobject $\overline{[\![\, \vec{x}, \vec{m} : \psi \,]\!]_{\mathbb{X}}}$ is in the image of $[\![\, - \,]\!]_{\mathbb{X}_{\vec{x}}}$. This is precisely the condition that $\mathbb{X}$ eliminates parameters, from which we deduce the result. ◻

Remark VII.30. Let $\mathbb{T}$ be a geometric theory over a signature Σ, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$ indexed by parameters $\mathfrak{K}$. As remarked in Remarks VII.7(iv), the condition that $\mathbb{X}$ eliminates parameters depends only on the signature of the theory $\mathbb{T}$. This can be retroactively justified topos-theoretically in light of Proposition VII.29 by equating those indexed model groupoids that eliminate parameters with those indexed model groupoids for which $\mathbf{p}^{\log}$ is an inclusion.

Let $\mathbb{E}_\Sigma$ denote the empty theory over the signature Σ. As $\mathbb{T}$ is a *quotient theory* of $\mathbb{E}_\Sigma$, the theory $\mathbb{T}$ is classified by a subtopos $\mathcal{E}_{\mathbb{T}} \rightarrowtail \mathcal{E}_{\mathbb{E}_\Sigma}$ (see [22, §3]), and so, by Proposition VII.29, $\mathbb{X}$ eliminates parameters if andonly if there are inclusions of subtopoi

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \rightarrowtail \mathcal{E}_{\mathbb{T}} \rightarrowtail \mathcal{E}_{\mathbb{E}_\Sigma},$$

whence we deduce that $\mathbb{X}$ eliminating parameters depended only on the signature Σ.

Since a geometric morphism is an equivalence of topoi if and only if it is both a surjection and an inclusion (see [63, Corollary A4.2.11]), we deduce the following:

Corollary VII.31. *Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for a geometric theory $\mathbb{T}$, and let $\mathbb{X}$ be indexed by a set of parameters $\mathfrak{K}$. The geometric morphism $\mathbf{p}^{\log}$ is an equivalence of topoi*

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if $\mathbb{X}$ is conservative and eliminates parameters.

Combining both Proposition VII.27 and Corollary VII.31 completes the proof of Theorem VII.8. Moreover, we can also use the results to deduce the sense in which the logical topologies are, essentially, the only topologies that need be considered for model groupoids.

Corollary VII.32. *Let $\mathbb{T}$ be a geometric theory, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$. If there exist topologies on X_0 and X_1 making $\mathbb{X}$ an open topological groupoid such that $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$, then there is an indexing of $\mathbb{X}$ by a set of parameters $\mathfrak{K}$ such that*

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \simeq \mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathbb{X}).$$

VII.5 Applications

In this section we present some applications of Theorem VII.8, divided as follows.

- The first two sections justify our use of indexings of models, i.e. presenting models as a subquotient of a set of parameters. Section VII.5.1 is devoted to the study of atomic theories. These are the only theories whose models of a representing groupoid may be indexed by disjoint sets of parameters. We will recover the logical 'topological Galois theory' result of [21] that the automorphism group of a single model represents an atomic theory if and only if the model is conservative and ultrahomogeneous. We also give a characterisation of Boolean topoi with enough points reminiscent of the characterisation established in [11].
- In Section VII.5.2, we demonstrate that, instead of subquotients, we can present the models in a representing groupoid as subsets of a set of parameters only in the case when the theory has decidable equality. We also generate examples of representing groupoids for decidable theories.
- We demonstrate in Section VII.5.3 that every representing model groupoid is Morita equivalent to its *étale completion*, i.e. the model groupoid with the same objects and all possible isomorphisms between these constituent models, paralleling the analogous result for localic groupoids (see [92, §7]). We also show that the étale completion of a representing model groupoid can be calculated as a topological closure in the fashion of [52, §4].
- We recover in Section VII.5.4 the representation theorems given by Butz and Moerdijk in [17] and by Awodey and Forssell in [5], [37] by demonstrating that the considered groupoids fall within a general framework of 'maximal groupoids'.
- Finally, having studied how to generate a representing groupoid of indexed models for any theory, we answer the converse direction and describe a theory which is represented by a given groupoid of indexed structures. This extends the techniques developed in [52, Theorem 4.14] for subgroups of the topological permutation group on a set.

VII.5.1 Atomic theories

In this section we will study those model groupoids of a theory that eliminate parameters when their constituent models are indexed by disjoint sets of parameters. We will observe in Proposition VII.35 that this requires the theory to be atomic.

We revisit Caramello's 'topological Galois theory' and demonstrate that the results of [21] concerning atomic theories can be recovered via the classification theorem. We also give a characterisation of Boolean topoi with enough points in a manner reminiscent to [11].

Definition VII.33 (Proposition D3.4.13 [63])**.** A geometric theory $\mathbb{T}$ is *atomic* if one of the following equivalent conditions is satisfied:

(i) for each context $\vec{x}$, $F^{\mathbb{T}}(\vec{x})$ is generated by its atoms,

(ii) the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ is *atomic* (see §C3.5 [63]).

If $\mathbb{T}$ is known to possess enough points, by [20, Theorem 3.16] we can add a further equivalent condition to the above list:

(iii) every (model-theoretic) type of $\mathbb{T}$ is *isolated*, also known as *principal* and *complete*, i.e. for each model M of $\mathbb{T}$ and tuple $\vec{n} \in M$, there is a formula $\chi_{\vec{n}}$, the *minimal formula* of $\vec{n}$, such that

   a) for any other tuple $\vec{n}'$ of the same sort as $\vec{n}$ in another model N of $\mathbb{T}$, then $N \vDash \chi_{\vec{n}}(\vec{n}')$ if and only if $\vec{n}$ and $\vec{n}'$ satisfy the same formulae – i.e. $\mathrm{tp}_M(\vec{n}) = \langle \chi_{\vec{n}} \rangle$, read as $\chi_{\vec{n}}$ *isolates* the type of $\vec{n}$ (see [89, §4.1]);
   b) for all formulae φ in context $\vec{x}$, either $\mathbb{T}$ proves the sequent $\chi_{\vec{m}} \vdash_{\vec{x}} \varphi$ or $\mathbb{T}$ proves $\chi_{\vec{m}} \wedge \varphi \vdash_{\vec{x}} \bot$ – equivalently, given a pair of tuples in two models, if $\mathrm{tp}_M(\vec{n}) \subseteq \mathrm{tp}_N(\vec{n}')$ then $\mathrm{tp}_M(\vec{n}) = \mathrm{tp}_N(\vec{n}')$.

Recall also from [63, Corollary C3.5.2] that, under the assumption that $\mathbb{T}$ has enough points, the properties that $\mathbb{T}$ is an atomic geometric theory and that $\mathbb{T}$ is a Boolean geometric theory (i.e. the classifying topos $\mathcal{E}_{\mathbb{T}}$ is Boolean) coincide.

Examples VII.34. (i) The terminology *minimal formula* is derived from the analogy with minimal polynomials. We define the theory of algebraically closed fields of finite characteristic which are algebraic over their prime subfield $\mathbb{ACF}^{\mathrm{alg}}_{\mathrm{fin}}$ as the theory for which, in addition to the usual axioms of an algebraically closed field, we also include as axioms the sequents

$$\top \vdash_{\emptyset} \bigvee_{p \text{ prime}} \underbrace{1 + 1 \cdots + 1}_{p \text{ times}} = 0,$$
$$\top \vdash_{x} \bigvee_{q \in \mathbb{Z}[x]} q(x),$$

in which the former expresses that the characteristic is finite while the latter expresses that the field is algebraic over its prime subfield.

This is an atomic theory with enough points. For each single element a in an algebraically closed field F algebraic over its prime subfield, the minimal formula of a is precisely the conjunction of the minimal polynomial of a with the formula $\top \vdash_{\emptyset} 1 + 1 \cdots + 1 = 0$ expressing the characteristic of the field. For a tuple $\vec{a} = (a_1, \ldots, a_n)$ of F, the minimal formula of $\vec{a}$ is the formula

$$\bigwedge_{i=1}^{n} q_i(x_1, \ldots, x_i) \wedge \underbrace{1 + 1 \cdots + 1}_{p \text{ times}} = 0,$$

where p is the characteristic of the field F, and $q_i(a_1, \ldots a_{i-1}, x_i)$ denotes the minimal polynomial of the element a_i over the field extension $F(a_1, \ldots, a_{i-1})$ (cf. Proposition VII.65).

(ii) The theory $\mathbb{D}_\infty$ of *infinite decidable objects* is also an atomic theory. It is the single-sorted theory with one binary relation $\neq$ and the axioms

$$x = y \wedge x \neq y \vdash_{x,y} \bot,$$
$$\top \vdash_{x,y} x = y \vee x \neq y,$$

and, for each $n \in \mathbb{N}$, the axiom

$$\top \vdash_x \exists y_1, \dots, y_n \bigwedge_{i \leqslant n} x \neq y_i \wedge \bigwedge_{i<j\leqslant n} y_i \neq y_j.$$

The minimal formula of a tuple $\vec{n}$ in a model is the formula

$$\bigwedge_{n_i = n_j} x_i = x_j \wedge \bigwedge_{n_i \neq n_j} x_i \neq x_j.$$

In a similar fashion, we can deduce that the theory of dense linear orders without endpoints $\mathbb{DLO}_\infty$, the theory of atomless Boolean algebras, and the theory of the Rado graph are all also atomic theories. A formal proof that these theories are atomic can be found in [18], [21].

Proposition VII.35. *Let $\mathbb{X}$ be model groupoid for $\mathbb{T}$ that is conservative and eliminates parameters for an indexing such that the set of parameters used to index each model $M \in X_0$ are mutually disjoint. Then $\mathbb{T}$ is an atomic theory.*

Proof. As each $M \in X_0$ is disjointly indexed from every other $N \in X_0$, the space $X_0^{\tau\text{-log}_0}$ is discrete. Hence, by [63, Lemma C3.5.3], $\mathbf{Sh}(X_0^{\tau\text{-log}_0})$ is an atomic topos. Recall from Lemma V.8 that there is an open surjective geometric morphism

$$\mathbf{Sh}(X_0^{\tau\text{-log}_0}) \longrightarrow \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \simeq \mathcal{E}_{\mathbb{T}},$$

and thus by applying, [63, Lemma C3.5.1] we obtain, the desired result. □

We now turn to the theory of [21] and consider the groupoid consisting of the automorphism group of a single model. We note that for this groupoid there is essentially only one indexing, the trivial indexing from Examples VII.3(i), since a parameter can be conflated with the element of the model it indexes. Thus, we will assume that the automorphism group is trivially indexed. We will show that, if $\mathbb{T}$ is an atomic theory with enough points, then the automorphism group of a single model eliminates parameters if and only if that model is ultrahomogeneous. Thus, we deduce the principal result of [21].

Elimination of parameters implies ultrahomogeneity. We first observe that elimination of parameters by the automorphism group of a single model implies ultrahomogeneity. Recall that the model M is *ultrahomogeneous* if each finite partial isomorphism

$$\begin{tikzcd} \vec{m} \arrow[r, "\sim"] \arrow[d, hook] & \vec{n} \arrow[d, hook] \\ M & M \end{tikzcd}$$

can be extended to a total isomorphism $M \xrightarrow{\alpha} M$.

Lemma VII.36. *If M is a model of an arbitrary geometric theory $\mathbb{T}$ such that the group* Aut(M) *eliminates parameters, then M is ultrahomogeneous.*

Proof. For a fixed tuple $\vec{m} \in M$, by hypothesis there is a formula without parameters such that
$$\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathrm{Aut}(M)} = [\![\vec{x} : \varphi]\!]_{\mathrm{Aut}(M)}.$$
If there is a partial isomorphism $\vec{m} \xrightarrow{\sim} \vec{n}$, then $\vec{m}, \vec{n} \in [\![\vec{x} : \varphi]\!]_{\mathrm{Aut}(M)}$. Therefore, $\vec{n}$ is an element of $\overline{[\![\vec{x} : \vec{x} = \vec{m}]\!]}_{\mathrm{Aut}(M)}$, and so there exists an automorphism $M \xrightarrow{\alpha} M$ such that $\alpha(\vec{m}) = \vec{n}$. □

Ultrahomogeneity and atomicity imply elimination of parameters. We now give the opposite implication that, under the assumption that $\mathbb{T}$ is atomic, if a model is ultrahomogeneous then its automorphism group eliminates parameters. Thus, by combining Lemma VII.36 above and Lemma VII.37 below, we recover the principal result of [21].

Lemma VII.37. *Let $\mathbb{T}$ be an atomic geometric theory with enough points. If M is a ultrahomogeneous model, then* $\mathrm{Aut}(M)$ *eliminates parameters.*

Proof. We claim that, for each tuple $\vec{m} \in M$,
$$\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathrm{Aut}(M)} = [\![\vec{x} : \chi_{\vec{m}}]\!]_{\mathrm{Aut}(M)},$$
where $\chi_{\vec{m}}$ is the minimal formula of $\vec{m}$. Since $M \vDash \chi_{\vec{m}}(\vec{m})$ by definition, one inclusion
$$\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathrm{Aut}(M)} \subseteq [\![\vec{x} : \chi_{\vec{m}}]\!]_{\mathrm{Aut}(M)}$$
is immediate by Remarks VII.7(v).

For the converse, if $\vec{m}' \in [\![\vec{x} : \chi_{\vec{m}}]\!]_{\mathrm{Aut}(M)}$, then $\vec{m}, \vec{m}'$ have the same type and so there is a partial isomorphism $\vec{m} \xrightarrow{\sim} \vec{m}'$. As M is ultrahomogeneous, this extends to a total automorphism $\alpha \colon M \to M$ for which $\alpha(\vec{m}) = \vec{m}'$. Hence, we obtain the converse inclusion
$$[\![\vec{x} : \chi_{\vec{m}}]\!]_{\mathrm{Aut}(M)} \subseteq \overline{[\![\vec{x} = \vec{m}]\!]}_{\mathrm{Aut}(M)}.$$
By Remarks VII.7(iii), this suffices to demonstrate that $\mathrm{Aut}(M)$ eliminates parameters. □

Corollary VII.38 (Theorem 3.1 [21])**.** *Let $\mathbb{T}$ be an atomic theory and let M be a model of $\mathbb{T}$. There is an equivalence of topoi*
$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{B}\mathrm{Aut}(M)$$
if and only if M is a conservative and ultrahomogeneous model. Here, the group $\mathrm{Aut}(M)$ *has been topologised with the* Krull topology *(also called the pointwise convergence topology). It is the coarsest topology making* $\mathrm{Aut}(M)$ *a topological group for which the subsets*
$$\left\{ M \xrightarrow{\alpha} M \,\middle|\, \alpha(\vec{n}) = \vec{n} \right\},$$
for every finite tuple $\vec{n} \in M$, form a basis of open neighbourhoods of the identity.

Examples VII.39 (§5 [21])**.** We revisit some of the examples of atomic theories from Examples VII.34 and describe conservative ultrahomogeneous models for them.

(a) Any infinite set, in particular $\mathbb{N}$, is a conservative ultrahomogeneous model for the theory $\mathbb{D}_\infty$ of infinite decidable objects. Therefore, $\mathbb{D}_\infty$ is classified by $\mathbf{B}\mathrm{Aut}(\mathbb{N})$, the *Schanuel topos*.

(b) The rationals $\mathbb{Q}$, with their usual ordering, is a conservative and ultrahomogeneous model for the theory $\mathbb{DLO}_\infty$. Its classifying topos is $\mathbf{B}\mathrm{Aut}(\mathbb{Q})$.

(c) Let $\mathcal{R}$ denote the Rado graph. It is a conservative and ultrahomogeneous model for its namesake theory, which is therefore classified by $\mathbf{B}\mathrm{Aut}(\mathcal{R})$.

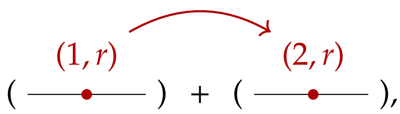
Recall from [20, Theorem 3.16] that if an atomic theory $\mathbb{T}$ is also *complete*, by which we mean that any model is conservative (or equivalently, by [20, Proposition 3.9], for every sentence φ, either $\mathbb{T}$ proves $\top \vdash \varphi$ or $\mathbb{T}$ proves $\varphi \vdash \bot$), then the theory $\mathbb{T}$ is also *countably categorical*, i.e. any two countable models of $\mathbb{T}$ are isomorphic. Therefore, it suffices in Corollary VII.38 to take M as the unique countable model of the theory – if this exists – since this will automatically also be an ultrahomogeneous model (see [52, §10.1]).

Example VII.40. Let $\mathbb{T}$ be an atomic theory and let M be a conservative model of $\mathbb{T}$. In order to assure the reader that ultrahomogeneity is a non-trivial requirement on M, despite the hefty conditions placed on $\mathbb{T}$ by being an atomic theory, we briefly describe a conservative model for $\mathbb{DLO}_\infty$ which is not ultrahomogeneous. Let $\mathbb{R}$ denote the real numbers with the usual ordering, and let $\mathbb{R} + \mathbb{R}$ denote the model whose underlying set is

$$\{1\} \times \mathbb{R} \cup \{2\} \times \mathbb{R}$$

given the lexicographic ordering. This is a conservative model of the theory of dense linear orders without endpoints (since this is a complete theory). However, it is not ultrahomogeneous.

We note that, for any $r \in \mathbb{R}$, the partial isomorphism $(1, r) \mapsto (2, r)$, as visualised in the diagram



cannot be extended to a total automorphism of $\mathbb{R} + \mathbb{R}$. If there did exist such a total automorphism of $\mathbb{R} + \mathbb{R}$, then the subset

$$\{1\} \times (-\infty, r) \subseteq \mathbb{R} + \mathbb{R},$$

being the down-segment of $(1, r)$, would necessarily be mapped isomorphically to the subset

$$\{1\} \times \mathbb{R} \cup \{2\} \times (-\infty, r) \subseteq \mathbb{R} + \mathbb{R},$$

the down-segment of $(2, r)$. However, the interval $(-\infty, r) \cong \{1\} \times (-\infty, r)$ is *Dedekind-complete*, meaning that every subset of $(-\infty, r)$ with an upper bound has a least upper bound, while $\{1\} \times \mathbb{R} \cup \{2\} \times (-\infty, r)$ is not Dedekind-complete, namely the subset $\{1\} \times \mathbb{R} \subseteq \{1\} \times \mathbb{R} \cup \{2\} \times (-\infty, r)$ does not have a least upper bound.

This one model therefore serves as a counterexample to several natural questions arising from the study of representing groupoids for topoi.

(i) Recall from [68] and [34] that a connected atomic topos is represented by the *localic* automorphism group of any of its points. This is because any point of a connected atomic topos is an open surjection (see [68, Proposition VII.4.1]). Being a complete atomic theory, the classifying topos for the theory $\mathbb{DLO}_\infty$ is a connected atomic topos. Thus, the theory is represented by the localic group of automorphisms $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ but not by the topological group of automorphisms of $\mathbb{R}+\mathbb{R}$ (see [21] or Corollary VII.38 above).

The discrepancy occurs because $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$, the localic automorphism group constructed in [34, Proposition 4.7], is not spatial. The underlying locale $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ of the localic automorphism group can be described as the classifying locale for the following geometric propositional theory.

a) For each pair of elements $x, y \in \mathbb{R}+\mathbb{R}$, we add a pair of basic propositions $[\alpha(x) = y]$ and $[x < y]$.

b) For every pair $x, y \in \mathbb{R}+\mathbb{R}$ with $x < y$, we add the sequent $\top \vdash [x < y]$ as an axiom to the propositional theory, and for every quadruple of elements $x, x', y, y' \in \mathbb{R}+\mathbb{R}$ where $x \neq x'$ and $y \neq y'$, we also add to our axioms the sequents

$$[\alpha(x) = y] \wedge [\alpha(x') = y] \vdash \bot,$$
$$[\alpha(x) = y] \wedge [\alpha(x) = y'] \vdash \bot$$

and

$$\top \vdash \bigvee_{x \in \mathbb{R}+\mathbb{R}} [\alpha(x) = y],$$
$$\top \vdash \bigvee_{y \in \mathbb{R}+\mathbb{R}} [\alpha(x) = y],$$

expressing that the symbol α encodes a bijection from $\mathbb{R}+\mathbb{R}$ to itself. Additionally, we include the bidirectional sequent

$$[\alpha(x) = y] \wedge [\alpha(x') = y'] \wedge [x < x'] \dashv\vdash [\alpha(x) = y] \wedge [\alpha(x') = y'] \wedge [y < y'],$$

as an axiom, expressing that α encodes an automorphism of $\mathbb{R}+\mathbb{R}$ as a linear order.

We note the similarities between this propositional theory, for which the locale $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ is its classifying locale, and the propositional theory $P[\mathbb{T}_{\cong}]$ from Section VI.3.1, for which $G_1^{\mathbb{T}}$ is its classifying locale.

A point $\alpha\colon \mathbf{2} \to \mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ of this locale evidently corresponds to an automorphism of the model $\mathbb{R}+\mathbb{R}$. We therefore deduce, by the above analysis, that the non-trivial open of $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ corresponding to the basic proposition $[(1, r) \mapsto (2, r)]$ is evaluated by α^{-1} as

$$\alpha^{-1}\left([(1, r) \mapsto (2, r)]\right) = \bot.$$

Hence, $\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}$ is not a spatial locale.

In summary, we have that

$$\mathcal{E}_{\mathbb{DLO}_\infty} \simeq \mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}} \not\simeq \mathbf{B}\mathrm{Pt}(\mathrm{Aut}(\mathbb{R}+\mathbb{R})^{\mathrm{loc}}) \simeq \mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R}).$$

It is then natural to wonder: if not $\mathbb{DLO}_\infty$, what theory is classified by the topos $\mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R})$? Such a theory is described in Example VII.61 as an application of the techniques exposited in Section VII.5.5.

(ii) Let $\mathbb{T}'$ be a theory of *presheaf type*, i.e. the classifying topos of $\mathbb{T}'$ is a presheaf topos. By [22, §6.1.1], this presheaf topos can be chosen to be the topos

$$\mathbf{Sets}^{\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Sets})^{\mathrm{op}}},$$

where f.p.$\mathbb{T}$-mod(**Sets**) denotes the category of *finitely presented models* (see [22, Definition 6.1.11]). We will say, in accordance with [18, Definition 2.3(a)], that a model M' of $\mathbb{T}'$ is *homogeneous with respect to the finitely presented models*, or *f.p.-homogeneous* for short, if for every pair of finitely presented models N, N' and homomorphisms $f : N \to M'$ and $g : N \to N'$ of $\mathbb{T}'$-models, there exists a homomorphism of $\mathbb{T}'$-models h such that the triangle

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "g"'] & M' \\ N' \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

commutes. Every ultrahomogeneous model is f.p.-homogeneous (see [18, Remark 2.4(a)]).

The theory of linear orders is a theory of presheaf type by [79, §VIII.8], and the finitely presented linear orders are simply the finite linear orders. Moreover, the dense linear orders without endpoints are precisely those linear orders that are f.p.-homogeneous (see [21, Remark 3.8(a)]). Thus, $\mathbb{R}+\mathbb{R}$ is an example of a f.p.-homogeneous model that is not ultrahomogeneous.

Boolean topoi with enough points. Extending Corollary VII.38, we can use the classification theorem to characterise Boolean topoi with enough points in a manner reminiscent of [11]. Recall that a topos with enough points is Boolean if and only if it is atomic.

We first require one lemma on the quotient theories of a theory classified by a Boolean topos. Recall from Definition III.48 that a quotient theory $\mathbb{T}'$ of $\mathbb{T}$ is a theory over the same signature whose axioms include the axioms of $\mathbb{T}$.

Lemma VII.41. *If $\mathbb{T}$ is a geometric theory whose classifying topos is Boolean, then every quotient theory of $\mathbb{T}$ is determined by the addition of a single extra sentence $\top \vdash_\emptyset \varphi$ as an axiom.*

Proof. There are two ways to see this: one topos-theoretic, and one syntactic. We include both, though of course they are merely translations of one another.

A quotient theory $\mathbb{T}'$ of $\mathbb{T}$ is classified by a subtopos of $\mathcal{E}_\mathbb{T}$. Every subtopos of a Boolean topos is *open* by Corollary 3.5 [58]. Therefore, being an open subtopos of $\mathcal{E}_\mathbb{T}$, $\mathcal{E}_{\mathbb{T}'}$ corresponds to a subterminal in $\mathcal{E}_\mathbb{T}$, i.e. a sentence $\{\,\emptyset : \varphi\,\}$.

Alternatively, if $\mathcal{E}_\mathbb{T}$ is a Boolean topos, then we recall from [63, §D3.4] that every formula of *infinitary first order logic* (i.e. including negation $\neg$, implication $\to$, universal quantification $\forall$, and infinitary conjunction $\bigwedge$) is $\mathbb{T}$-provably equivalent to a geometric formula. This is equivalent to requiring that the doctrine $F^{\mathbb{T}} \colon \mathbf{Con}_N \to \mathbf{Frm}_{\text{open}}$ associated to the theory factors through complete Boolean algebras **CBool**. We are therefore free to manipulate geometric sequents as though they existed in infinitary first order logic. Hence, we easily see that any quotient theory $\mathbb{T}' = \mathbb{T} \cup \left\{ \varphi_i \vdash_{\vec{x}_i} \psi_i \,\middle|\, i \in I \right\}$ of $\mathbb{T}$ is equivalent to the theory

$$\mathbb{T}' \equiv_s \mathbb{T} \cup \left\{ \top \vdash_{\emptyset} \forall \vec{x}_i\, \varphi_i \to \psi_i \,\middle|\, i \in I \right\} \equiv_s \mathbb{T} \cup \left\{ \top \vdash_{\emptyset} \bigwedge_{i \in I} \forall \vec{x}_i\, \varphi_i \to \psi_i \right\},$$

i.e. the quotient of $\mathbb{T}$ by a single sentence. □

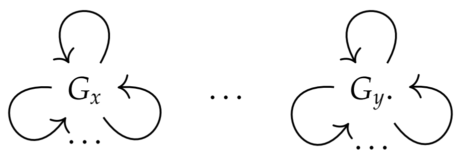
Corollary VII.42. *A topos $\mathcal{E}$ with enough points is Boolean if and only if there is a set of topological groups $\{ G_x \mid x \in X_0 \}$ such that*

$$\mathcal{E} \simeq \coprod_{x \in X_0} \mathbf{B}G_x.$$

Proof. Firstly, we recognise that a topos of the form $\coprod_{x \in X_0} \mathbf{B}G_x$ is really just the topos of sheaves for the topological groupoid

$$\mathbb{G} = \left(\coprod_{x \in X_0} G_x \rightrightarrows \coprod_{x \in X_0} 1 \cong X_0^{\delta} \right).$$

This topological groupoid is, of course, automatically open. We could call groupoids of this form the *bouquet* groupoids since, when written out diagrammatically, they appear as a collection of 'flowers' – the group elements $g \in G_x$ being the 'petals', e.g.



By Proposition VII.35, if there is an equivalence $\mathcal{E} \simeq \coprod_{x \in X_0} \mathbf{B}G_x$ for some set of topological groups $\{ G_x \mid x \in X_0 \}$, then $\mathcal{E}$ is Boolean.

For the converse direction, let $\mathbb{T}$ be a geometric theory classified by the topos $\mathcal{E}$. By the hypotheses, $\mathbb{T}$ is an atomic theory with enough points. We can therefore find a conservative set of models X_0 for $\mathbb{T}$. In fact we can choose each model $M \in X_0$ to be ultrahomogeneous since, via a standard result in model theory, every model is an elementary substructure of a ultrahomogeneous model (see [52, §10.2]). Moreover, we can evidently choose the models $M \in X_0$ to be pairwise elementarily inequivalent (i.e. no two models satisfy all the same sentences) and also pairwise disjoint. Therefore, by Lemma VII.37 we deduce that each automorphism group $\mathrm{Aut}(M)$ eliminates parameters.

Thus, when given the trivial indexing, the model groupoid

$$\coprod_{M \in X_0} \mathrm{Aut}(M),$$

obtained by taking as objects the models $M \in X_0$ and as arrows all automorphisms, also eliminates parameters.

To see this, we first note that for each definable with parameters

$$[\![\vec{x}, \vec{m} : \psi]\!]_{\coprod_{M\in X_0} \mathrm{Aut}(M)},$$

the parameters $\vec{m}$ are only instantiated in one model $M \in X_0$ (since the models of X_0 were chosen to be pairwise disjoint). Therefore, since there are no arrows between distinct models of our groupoid, (modulo a transparent abuse of notation) we have that

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]}_{\coprod_{M\in X_0} \mathrm{Aut}(M)} = \overline{[\![\vec{x}, \vec{m} : \psi]\!]}_{\mathrm{Aut}(M)}.$$

Let φ be a formula without parameters such that

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]}_{\mathrm{Aut}(M)} = [\![\vec{x} : \varphi]\!]_{\mathrm{Aut}(M)}.$$

We are not quite done since $[\![\vec{x} : \varphi]\!]_{\mathrm{Aut}(M)} \neq [\![\vec{x} : \varphi]\!]_{\coprod_{M\in X_0} \mathrm{Aut}(M)}$. Instead, we must find a formula that isolates those realisations of φ in M from those in other models $M' \in X_0$. This is achieved by Lemma VII.41. Let $\mathbb{T}_M$ denote the *theory of the model M*, i.e. the set of geometric sequents

$$\mathbb{T}_M = \left\{ \chi \vdash_{\vec{x}} \xi \,\middle|\, [\![\vec{x} : \chi]\!]_M \subseteq [\![\vec{x} : \xi]\!]_M \right\}.$$

This is evidently a quotient theory of $\mathbb{T}$. Thus, by Lemma VII.41, there exists a sentence ξ_M such that M is the only model in X_0 which satisfies ξ_M – since the models of X_0 were chosen to be pairwise elementarily inequivalent. Therefore, we have that

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]}_{\coprod_{M\in X_0} \mathrm{Aut}(M)} = [\![\vec{x} : \varphi \wedge \xi_M]\!]_{\coprod_{M\in X_0} \mathrm{Aut}(M)}$$

as required. Thus, $\coprod_{M\in X_0} \mathrm{Aut}(M)$ is a conservative model groupoid for $\mathbb{T}$ that eliminates parameters and so, by Proposition VII.29, we conclude that $\mathcal{E} \simeq \coprod_{M\in X_0} \mathbf{B}\mathrm{Aut}(M)$, once each automorphism group has been suitably topologised. □

Example VII.43 (Proposition 2.4 [25])**.** We return to the theory $\mathbb{ACF}^{\mathrm{alg}}_{\mathrm{fin}}$ defined in Examples VII.34(i). Being an atomic theory with enough points, by Corollary VII.42 we know it can be presented as a coproduct of topoi of actions by topological groups. Indeed, the theory is classified by the topos

$$\coprod_{p \text{ prime}} \mathbf{B}\mathrm{Aut}\left(\overline{\mathbb{Z}/\langle p\rangle}\right),$$

where $\overline{\mathbb{Z}/\langle p\rangle}$ is the algebraic closure of $\mathbb{Z}/\langle p\rangle$, and $\mathrm{Aut}\left(\overline{\mathbb{Z}/\langle p\rangle}\right)$ has been topologised with the usual Krull topology. This is precisely [25, Proposition 2.4].

The principal result of [11] classifies Boolean coherent topoi. As coherent topoi automatically have enough points, the result can be obtained from Corollary VII.42 by discerning when the topos $\coprod_{x\in X_0} \mathbf{B}G_x$ is coherent. This occurs when X_0 is a finite set and each G_x is a *coherent* topological group. We refer to [11] for the details.

VII.5.2 Decidable theories

We saw in Proposition VII.35 that we cannot, in general, require that the models in a representing model groupoid be disjointly indexed. In this subsection, we demonstrate further that nor can we remove the requirement that multiple parameters may index the same element of a model, i.e. that each model is presented as a subquotient of its set of parameters rather than a subset. We will observe that this is possible only if the theory is decidable.

Definition VII.44. A geometric theory $\mathbb{T}$ over a signature Σ is *decidable* if, for each pair of free variables x, x' of the same sort of Σ, there is a formula in context x, x', which we suggestively denote as $x \neq x'$, such that $\mathbb{T}$ proves the sequents

$$x = x' \wedge x \neq x' \vdash_{x,x'} \bot, \quad \top \vdash_{x,x'} x = x' \vee x \neq x'.$$

Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for a geometric theory $\mathbb{T}$. Consider the trivial indexing of $\mathbb{X}$, described in Examples VII.3(i), with parameters as the elements of the constituent models $\bigcup_{M \in X_0} M$. For this indexing, each element $n \in M \in X_0$ is indexed by precisely one parameter.

It is not hard to see that, up to isomorphism, this is the unique indexing of $\mathbb{X}$ with this property. Suppose that $\mathbb{X}$ is indexed by a set of parameters $\mathfrak{K}$ in such a way that, for each $n \in M \in X_0$, there is a unique parameter $m \in \mathfrak{K}$ that indexes n. This is equivalent to presenting each model $M \in X_0$ as a subset of the set of parameters, rather than a subquotient. By replacing the underlying set of M with the corresponding subset of parameters, we have now trivially indexed $\mathbb{X}$ (we have ignored the fact that our set of parameters may include some unused parameters).

Proposition VII.45. *Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a conservative model groupoid for a theory $\mathbb{T}$. If, when $\mathbb{X}$ is given the trivial indexing, $\mathbb{X}$ eliminates parameters, then $\mathbb{T}$ is a decidable theory.*

Proof. We must show that, for each pair of free variables x, x' of the same sort, the formula $x = x'$ has a complement in $F^{\mathbb{T}}(x, x')$. As $\mathbb{X}$ is conservative and eliminates parameters it is a representing model groupoid by Theorem VII.8, and so by the isomorphism

$$F^{\mathbb{T}}(x, x') \cong \mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)}(\llbracket x, x' : \top \rrbracket_{\mathbb{X}}),$$

finding a complement for the formula $x = x'$ is equivalent to showing that $\llbracket x = x' \rrbracket_{\mathbb{X}}$ has a complement.

We claim that this complement, which we denote by $\llbracket x \neq x' \rrbracket_{\mathbb{X}}$, is given by

$$\bigcup_{\substack{m,m' \in \bigcup_{M \in X_0} M \\ m \neq m'}} \llbracket x = m \wedge x' = m' \rrbracket_{\mathbb{X}}.$$

We must first show that $\llbracket x \neq x' \rrbracket_{\mathbb{X}}$ does indeed define a subobject, i.e. a stable open subset, of $\llbracket x, x' : \top \rrbracket_{\mathbb{X}}$. The subset $\llbracket x \neq x' \rrbracket_{\mathbb{X}}$ is the union of opens, and therefore open itself. Now suppose we are given an element $\langle n, n', M \rangle \in \llbracket x \neq x' \rrbracket_{\mathbb{X}}$. By the definition, $n \neq n'$ and so $\alpha(n) \neq \alpha(n')$ for any isomorphism $M \xrightarrow{\alpha} N$. Therefore, $\langle \alpha(n, n'), N \rangle$ is also an element of $\llbracket x \neq x' \rrbracket_{\mathbb{X}}$. Thus, $\llbracket x \neq x' \rrbracket_{\mathbb{X}}$ is a stable open as desired.

It is now not hard to conclude that $[\![\, x \neq x' \,]\!]_{\mathbb{X}}$ is a complement to $[\![\, x = x' \,]\!]_{\mathbb{X}}$. Given $\langle n, n', M\rangle \in [\![\, x, x' : \top \,]\!]_{\mathbb{X}}$, either $n = n'$ or $n \neq n'$, yielding

$$[\![\, x = x' \,]\!]_{\mathbb{X}} \cup [\![\, x \neq x' \,]\!]_{\mathbb{X}} = [\![\, x, x' : \top \,]\!]_{\mathbb{X}}.$$

It is similarly easy to conclude that $[\![\, x = x' \,]\!]_{\mathbb{X}} \cap [\![\, x \neq x' \,]\!]_{\mathbb{X}} = \emptyset$. □

Since not every geometric theory is decidable, we cannot in general require that each element of a model groupoid is indexed by a unique parameter. Even when our theory is decidable, we must allow for the models of our groupoid to share elements, i.e. parameters.

Examples of representing groupoids for decidable theories. Finally, we present a useful result from which it is possible to easily generate representing groupoids for many theories. Note that the only theories that can be represented by the groupoids generated using the below method must also be decidable. This can be seen by an application of Proposition VII.45.

Proposition VII.46. *Let $\mathbb{T}$ be geometric theory over a signature Σ. Suppose that a conservative set of models for $\mathbb{T}$ can be found as substructures of an ultrahomogeneous Σ-structure U whose theory $\mathrm{Th}(U)$ (i.e. the theory over Σ whose axioms are all sequents satisfied by U) is atomic, and moreover the minimal formula $\chi_{\vec{n}}$ of any tuple of elements $\vec{n} \in U$ is quantifier free. Then the model groupoid $\mathrm{Sub}_{\mathbb{T}}(U)$ of $\mathbb{T}$,*

(i) whose objects are the substructures of U that are models of $\mathbb{T}$,

(ii) and whose arrows are all isomorphisms between these,

is a representing model groupoid for $\mathbb{T}$, i.e. there is an indexing of $\mathrm{Sub}_{\mathbb{T}}(U)$ such that

$$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}\left(\mathrm{Sub}_{\mathbb{T}}(U)^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right).$$

Proof. By hypothesis, the model groupoid $\mathrm{Sub}_{\mathbb{T}}(U)$ is conservative. It remains to show that the the groupoid $\mathrm{Sub}_{\mathbb{T}}(U)$ has an indexing by a set of parameters for which the groupoid eliminates parameters. The indexing set we use are the elements of the Σ-structure U. The indexing of a model $M \in \mathrm{Sub}_{\mathbb{T}}(U)$ is determined by the inclusion $M \subseteq U$ of M as a substructure of U.

Since U is an ultrahomogeneous model of the atomic theory $\mathrm{Th}(U)$, we have that, by Lemma VII.37, for each tuple $\vec{m} \in U$,

$$\overline{[\![\, \vec{x} = \vec{m} \,]\!]}_{\mathrm{Aut}(U)} = [\![\, \vec{x} : \chi_{\vec{m}} \,]\!]_{\mathrm{Aut}(U)},$$

where $\chi_{\vec{m}}$ is the minimal formula of $\vec{m} \in U$.

We claim that

$$\overline{[\![\, \vec{x} = \vec{m} \,]\!]}_{\mathrm{Sub}_{\mathbb{T}}(U)} = [\![\, \vec{x} : \chi_{\vec{m}} \,]\!]_{\mathrm{Sub}_{\mathbb{T}}(U)}.$$

Since $\chi_{\vec{m}}$ is a quantifier free formula, $M \vDash \chi_{\vec{m}}(\vec{m})$ for any $\mathbb{T}$-model $M \subseteq U$ that contains $\vec{m}$. Thus, the first inclusion

$$\overline{[\![\, \vec{x} = \vec{m} \,]\!]}_{\mathrm{Sub}_{\mathbb{T}}(U)} \subseteq [\![\, \vec{x} : \chi_{\vec{m}} \,]\!]_{\mathrm{Sub}_{\mathbb{T}}(U)}$$

follows from Remarks VII.7(v). Conversely, given another witness $\vec{n}$ of $\chi_{\vec{m}}$ in a $\mathbb{T}$-model $N \subseteq U$, there exists, by the ultrahomogeneity of U, an automorphism α of U that sends $\vec{m}$ to $\vec{n}$. We have that $\alpha^{-1}(N), N \subseteq U$ constitute a pair of $\mathbb{T}$-models, and

$$\alpha|_{\alpha^{-1}(N)} \colon \alpha^{-1}(N) \to N$$

is an isomorphism of Σ-structures that sends $\vec{m} \in \alpha^{-1}(N)$ to $\vec{n} \in N$. Therefore, we have that $\langle \vec{n}, N \rangle \in \overline{[\![\vec{x} = \vec{m}]\!]}_{\mathrm{Sub}_{\mathbb{T}}(U)}$, completing the reverse inclusion. □

Examples VII.47. We apply Proposition VII.46 to give a pair of simple examples of model groupoids for decidable theories.

(i) (§2.4.1 [5]) The theory of *decidable objects* is the single-sorted theory with one binary predicate $\neq$ and the axioms

$$x = x' \wedge x \neq x' \vdash_{x,x'} \bot, \quad \top \vdash_{x,x'} x = x' \vee x \neq x'.$$

As observed in Examples VII.39(a), the natural numbers is an ultrahomogeneous model of the theory of decidable objects. The theory of $\mathbb{N}$ is the theory of infinite decidable objects $\mathbb{D}_\infty$ from Examples VII.34(ii), an atomic theory whose minimal formulae are quantifier free.

Moreover, the subsets of $\mathbb{N}$ are a conservative set of models for the theory of decidable objects. Hence, by an application of Proposition VII.46, the theory of decidable objects is classified by

$$\mathbf{Sh}\left(\mathrm{Sub}(\mathbb{N})^{\tau\text{-}\mathrm{log}_1}_{\tau\text{-}\mathrm{log}_0}\right).$$

(ii) Let K be a field. We denote by $\mathbb{T}_{(-/K)}$ the theory of *algebraic extensions* of K. This is the single-sorted theory over the signature consisting of the standard signature of a ring with an additional constant symbol for each element of K. The axioms of $\mathbb{T}_{(-/K)}$ consist of the following:

a) the standard axioms of a field and an axiom $\top \vdash \varphi(\vec{k})$ for each sentence φ with constants $\vec{k} \in K$ satisfied by K, ensuring that each model of $\mathbb{T}_{(-/K)}$ is a field extension of K,

b) and the sequent

$$\top \vdash_x \bigvee_{q \in K[x]} q(x) = 0,$$

expressing that any model is an algebraic extension of K.

The algebraic closure $\overline{K}$ of K is an ultrahomogeneous structure whose theory is atomic and whose minimal formulae are quantifier free (cf. Examples VII.34(i)). Hence, by an application of Proposition VII.46, the theory $\mathbb{T}_{(-/K)}$ is classified by the topos

$$\mathbf{Sh}\left(\mathrm{Sub}\left(\overline{K}\right)^{\tau\text{-}\mathrm{log}_1}_{\tau\text{-}\mathrm{log}_0}\right),$$

where $\mathrm{Sub}\left(\overline{K}\right)$ is the groupoid of intermediate extensions of K, and all isomorphisms between these.

By Proposition VII.45, the theory $\mathbb{T}_{(-/K)}$ is decidable. Indeed, we can identify the complement of the equality predicate as the formula $\exists y \; y \cdot (x - x') = 1$.

VII.5.3 Étale complete groupoids

We now study the behaviour of a representing model groupoid when we expand its set of arrows, inspired by the consequence of the descent theory of Joyal and Tierney that every open localic groupoid is *Morita equivalent* to its *étale completion* (see [92, Definition 7.2] and Remark VI.6 above). Here we study the equivalent topological definition.

Definitions VII.48 (cf. Definition 7.2 [92])**.** Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for $\mathbb{T}$.

(i) The groupoid $\mathbb{X}$ is said to be *étale complete* if every $\mathbb{T}$-model isomorphism between models $M, N \in X_0$ is instantiated in X_1.

(ii) We denote by $\hat{\mathbb{X}}$ the *étale completion* of $\mathbb{X}$. This is the model groupoid $\hat{\mathbb{X}}$ whose set of objects is the same set of models X_0 as for $\mathbb{X}$, but whose arrows are all $\mathbb{T}$-model isomorphisms between models $M, N \in X_0$.

So far in Section VII.5.2 and Section VII.5.1, the only specific examples of representing groupoids we have considered have all been étale complete model groupoids. Our classification given in Theorem VII.8 is powerful enough to also recognise a representing model groupoid even when it is not étale complete. We give an example of a representing model groupoid that is étale incomplete in Example VII.49. However, we will observe that, just as is the case for localic groupoids, every open topological groupoid is *Morita equivalent* to its étale completion, by which we mean that their topoi of equivariant sheaves are equivalent.

Example VII.49. Let $\mathbb{T}$ be an atomic theory and M a ultrahomogeneous and conservative model of $\mathbb{T}$. We note that we do not require all the automorphisms of M in order to extend every possible partial isomorphism of finite substructures. Therefore, by taking a certain subgroup of $\mathrm{Aut}(M)$, we would still be able to use the ultrahomogeneity property that was so crucial in proving that $\mathrm{Aut}(M)$ eliminates imaginaries in Lemma VII.37. We manufacture such an example below.

We once again consider the theory $\mathbb{DLO}_\infty$. Recall from Examples VII.39 that this theory is represented by the automorphism group $\mathrm{Aut}(\mathbb{Q})$. We will show that we can take a (topologically dense) subgroup $\mathbb{X}$ of $\mathrm{Aut}(\mathbb{Q})$ which does not contain all automorphisms, and yet $\mathbb{X}$ eliminates parameters and hence is a representing group.

We note that, for any rational number $r \in \mathbb{Q}$, the map $p \mapsto p + r$ is an automorphism of $\mathbb{Q}$. We will say that an automorphism $\alpha \colon \mathbb{Q} \to \mathbb{Q}$ is *boundedly additive* if, apart from a bounded interval, it is given by addition. Explicitly, α is boundedly additive if there are bounded (closed) intervals $[q_1, r_1] \subseteq \mathbb{Q}$ and $[q_2, r_2] \subseteq \mathbb{Q}$ such that:

(i) firstly, α maps $[q_1, r_1]$ to $[q_2, r_2]$,

(ii) on the interval $(-\infty, q_1)$, α acts by $p \mapsto p + q_2 - q_1$,

(iii) and on the interval (r_1, ∞), α acts by $p \mapsto p + r_2 - r_1$.

The identity is clearly boundedly additive, and if α and γ are boundedly additive, then by choosing a sufficiently large interval we can ensure that their composite $\alpha \circ \gamma$ is boundedly additive too. Let $\mathbb{X}$ denote the subgroup of $\mathrm{Aut}(\mathbb{Q})$ of boundedly additive automorphisms.

We claim that for any tuple $\vec{q}_1 \in \mathbb{Q}$, we have that

$$\overline{[\![\vec{y} = \vec{q}_1]\!]_{\mathbb{X}}} = [\![\vec{y} : \chi_{\vec{q}_1}]\!]_{\mathbb{X}},$$

where $\chi_{\vec{q}_1}$ is the minimal formula of $\vec{q}_1$, and thus that $\mathbb{X}$ eliminates parameters. We automatically have one inclusion

$$\overline{[\![\vec{y} = \vec{q}_1]\!]_{\mathbb{X}}} \subseteq [\![\vec{y} : \chi_{\vec{q}_1}]\!]_{\mathbb{X}}.$$

For the converse, we must show that for any other tuple $\vec{q}_2 \in \mathbb{Q}$ with the same order type as $\vec{q}_1$, there is a boundedly additive automorphism $\alpha \colon \mathbb{Q} \to \mathbb{Q}$ that maps $\vec{q}_1$ onto $\vec{q}_2$. This is straightforward. Let q_1 and r_1 denote, respectively, the least and greatest elements of $\vec{q}_1$, and similarly define q_2, r_2 for $\vec{q}_2$. Using the standard back-and-forth methods one uses to show that $\mathbb{Q}$ is ultrahomogeneous (see [52, §3.2]), we can construct an order isomorphism $[q_1, r_1] \cong [q_2, r_2]$ that maps $\vec{q}_1$ onto $\vec{q}_2$. It is now clear that this can be extended to a total and boundedly additive automorphism of $\mathbb{Q}$. Thus, $\mathbb{X}$ is an automorphism subgroup on a conservative model that eliminates parameters, and hence a representing group of the theory.

The subgroup $\mathbb{X}$ is not the whole group $\mathrm{Aut}(\mathbb{Q})$. An example of an automorphism of $\mathbb{Q}$ that is not boundedly additive can be constructed out of one which is. Firstly we note that $\mathbb{Q}$ is order isomorphic to countably many copies of itself given the lexicographic ordering since, given some irrational a,

$$\mathbb{Q} \cong \bigcup_{n \in \mathbb{Z}} (a+n, a+n+1) \cong \coprod_{\omega_0} \mathbb{Q}.$$

Let α be a boundedly additive automorphism whose non-additive part $[q_1, r_1] \cong [q_2, r_2]$ is truly non-additive (the automorphism α could be, for example, the total automorphism induced by the partial isomorphism $1 < 2 < 4 \mapsto 1 < 3 < 4$). An automorphism of $\mathbb{Q}$ which is not boundedly additive is now obtained via the composite

$$\mathbb{Q} \cong \coprod_{\omega_0} \mathbb{Q} \xrightarrow{\coprod_{\omega_0} \alpha} \coprod_{\omega_0} \mathbb{Q} \cong \mathbb{Q}.$$

Proposition VII.50. *Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for a geometric theory $\mathbb{T}$, indexed by a set of parameters $\mathfrak{K}$, that is conservative and eliminates parameters. For any other model groupoid $\mathbb{X}' = (X_1' \rightrightarrows X_0')$ such that $X_0 = X_0'$ and $X_1 \subseteq X_1'$, i.e. $\mathbb{X}$ is a surjective on objects subgroupoid of $\mathbb{X}'$, then $\mathbb{X}'$ is also a conservative model groupoid that eliminates parameters when the models $M \in X_0 = X_0'$ are given the same indexing by $\mathfrak{K}$.*

Proof. We first note that, since $\mathbb{X}$ and $\mathbb{X}'$ contain as objects the same models given the same indexing by the parameters $\mathfrak{K}$, for a formula ψ and a tuple of parameters $\vec{m}$, we have that $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}} = [\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}'}$. We also conclude that $\mathbb{X}'$ is conservative since $\mathbb{X}$ is.

Let $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ be a definable with parameters. Since $\mathbb{X}$ eliminates parameters, there is a formula φ such that

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

We claim that $\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}'}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}'}$ too. One inclusion is immediate since

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}'} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}} = \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} \subseteq \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}'}}.$$

For the converse inclusion, for each element $\langle \vec{n}, N \rangle \in \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}'}}$, there exists some model M and $\vec{n}'$ such that $M \vDash \psi(\vec{n}', \vec{m})$ and a $\mathbb{T}$-model isomorphism $M \xrightarrow{\alpha} N \in X_1'$ such that $\alpha(\vec{n}') = \vec{n}$. Hence, $M \vDash \varphi(\vec{n}')$ since

$$\left\langle \vec{n}', M \right\rangle \in [\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}} \subseteq \overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}},$$

and so $N \vDash \varphi(\vec{n})$ too. □

Hence, we are immediately able to deduce the following:

Corollary VII.51. *If $\mathbb{X} = (X_1 \rightrightarrows X_0)$ is a representing model groupoid for $\mathbb{T}$, then $\mathbb{X}$ is Morita equivalent to its étale completion, i.e. there exists an indexing of the models $M \in X_0$ such that*

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \simeq \mathbf{Sh}\left(\hat{\mathbb{X}}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right).$$

The étale completion as the topological closure. The étale completion of a model groupoid can be calculated entirely topologically via an adaptation of [52, Theorem 4.14]. Therein, it is demonstrated that, for a subgroup $G \subseteq \mathbf{Sym}(A)$ of the permutation group on a set A, the following are equivalent.

(i) The subgroup $G \subseteq \mathbf{Sym}(A)$ is a closed set, when $\mathbf{Sym}(A)$ is endowed with the Krull topology (also called the pointwise convergence topology).

(ii) The group G is the automorphism group of the set A when equipped with a Σ-structure, for some single-sorted signature Σ.

We present how this result can be adapted to calculate the étale completion.

Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a model groupoid for a geometric theory $\mathbb{T}$ over a signature Σ with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$ by a set of parameters $\mathfrak{K}$. For each pair $M, N \in X_0$, we define the *hom-space* $\mathrm{Hom}_{\mathbb{X}}(M, N)$ as the subspace

$$\mathrm{Hom}_{\mathbb{X}}(M, N) = s^{-1}(M) \cap t^{-1}(N) \subseteq X_1^{\tau\text{-log}_1}.$$

Equivalently, $\mathrm{Hom}_{\mathbb{X}}(M, N)$ is the set of isomorphisms $M \xrightarrow{\alpha} N \in X_1$ endowed with the topology generated by the basis

$$\left[\!\left[\vec{b} \mapsto \vec{c} \right]\!\right]_{\mathbb{X}(M,N)} = \left\{ M \xrightarrow{\alpha} N \,\middle|\, \alpha(\vec{b}) = \vec{c} \right\},$$

for each pair of tuples of parameters $\vec{b}, \vec{c} \in \mathfrak{K}$.

If we were to forget that the models M and N had Σ-structure, we could still construct a hom-space $\mathbf{Iso}[M, N]$ of *all* isomorphisms between the underlying sets interpreting the sorts of M, N. The space $\mathbf{Iso}[M, N]$ is endowed with the analogous topology generated by the basis

$$\left[\!\left[\vec{b} \mapsto \vec{c} \right]\!\right]_{\mathbf{Iso}[M,N]} = \left\{ M \xrightarrow{\alpha} N \,\middle|\, \alpha(\vec{b}) = \vec{c} \right\},$$

for each pair of tuples of parameters $\vec{b}, \vec{c} \in \mathfrak{K}$. Evidently, $\mathrm{Hom}_{\mathbb{X}}(M, N)$ can be embedded as a subspace into $\mathbf{Iso}[M, N]$.

Proposition VII.52 (Theorem 4.14 [52]). *Suppose that $\mathbb{X}$ eliminates parameters. For each pair $M, N \in X_0$, the hom-space* $\mathrm{Hom}_{\hat{\mathbb{X}}}(M, N)$ *in the étale completion $\hat{\mathbb{X}}$ is the topological closure of the subspace*

$$\mathrm{Hom}_{\mathbb{X}}(M, N) \subseteq \mathbf{Iso}[M, N].$$

Proof. We must show that a point $M \xrightarrow{\alpha} N \in \mathbf{Iso}[M, N]$ is an accumulation point of $\mathrm{Hom}_{\mathbb{X}}(M, N)$ if and only if it is an isomorphism of M and N as Σ-structures.

First, suppose that α preserves Σ-structure, and let $\left[\!\left[\vec{b} \mapsto \vec{c} \right]\!\right]_{\mathbf{Iso}[M,N]}$ be any basic open neighbourhood of α. Since $\mathbb{X}$ eliminates parameters, there exists a formula χ over Σ such that

$$\overline{\left[\!\left[\vec{x} = \vec{b} \right]\!\right]_{\mathbb{X}}} = [\![\vec{x} : \chi]\!]_{\mathbb{X}} = [\![\vec{x} : \chi]\!]_{\hat{\mathbb{X}}}.$$

The isomorphism α preserves the interpretation of χ, and so

$$\alpha(\vec{b}) = \vec{c} \in [\![\vec{x} : \chi]\!]_{\hat{\mathbb{X}}} = \overline{\left[\!\left[\vec{x} = \vec{b} \right]\!\right]_{\mathbb{X}}}.$$

Therefore, there exists an isomorphism

$$M \xrightarrow{\gamma} N \in \mathrm{Hom}_{\mathbb{X}}(M, N) \subseteq \mathbf{Iso}[M, N]$$

such that $\gamma \in [\![\vec{b} \mapsto \vec{c}]\!]_{\mathbf{Iso}[M,N]}$. Hence, α is an accumulation point of $\mathrm{Hom}_{\mathbb{X}}(M, N)$.

Conversely, if α is an accumulation point of $\mathrm{Hom}_{\mathbb{X}}(M, N)$, then for each tuple of parameters $\vec{m} \in \mathfrak{K}$, there is an isomorphism of Σ-structures $M \xrightarrow{\gamma} N$ such that $\alpha(\vec{m}) = \gamma(\vec{m})$. Thus, since every tuple of elements of M is the interpretation of some tuple of parameters, α preserves the Σ-structure. □

VII.5.4 Maximal groupoids

We now turn to the topological groupoids considered in the works of Awodey, Butz, Forssell and Moerdijk [5], [17], [37] and demonstrate that these too fall within our general framework. Let $\mathbb{X}$ be a model groupoid for a geometric theory $\mathbb{T}$ indexed by a set of parameters $\mathfrak{K}$. Recall from Remarks VII.7(v) that if $\mathbb{X}$ is conservative and eliminates parameters, then for every tuple of parameters $\vec{m} \in \mathfrak{K}$, the formula in context

$$\left\{ \vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right\}$$

can be thought of as a 'universal upper bound' for elimination of parameters in that we always have an inclusion

$$\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}} \subseteq \left[\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\right]_{\mathbb{X}}.$$

The particular model groupoids $\mathbb{X}$ considered in [5], [17], [37] can be considered to be *maximal* in the sense that this inclusion is an equality

$$\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}} = \left[\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\right]_{\mathbb{X}}.$$

As observed in Section VI.3.3, the topological groupoids considered in [5], [17], [37] are also closely related to the original Joyal-Tierney representation result [68, Theorem VIII.3.2] recalled in Section VI.2.

We briefly motivate the use of what we will call *Forssell groupoids*. If we were to equate open representing groupoids of a theory $\mathbb{T}$ with those model groupoids from which all other models can be reconstructed, then intuitively the groupoid of *all* models contains 'sufficient information'. However, since a theory can have unboundedly many models, this is not a small groupoid. We might imagine that it suffices to restrict to the groupoid of all models of some sufficiently large cardinality. That this is the case is demonstrated in [5], [36], [37].

Definition VII.53 (§1.2 [5], §3.1 [37])**.** Let $\mathbb{T}$ be a geometric theory and let $\mathfrak{K}$ be an infinite set. The *Forssell groupoid* $\mathcal{FG}(\mathfrak{K})$ is the étale complete groupoid of all models whose underlying sets are subquotients of $\mathfrak{K}$, i.e. the groupoid of all $\mathfrak{K}$-indexed models.

If $\mathbb{T}$ is a geometric theory whose $\mathfrak{K}$-indexed models are conservative, then the groupoid $\mathcal{FG}(\mathfrak{K})$ is an open representing groupoid. Thus, by the classification in Theorem VII.8, we know that there exists an indexing of $\mathcal{FG}(\mathfrak{K})$ for which the groupoid eliminates parameters. Given the construction of $\mathcal{FG}(\mathfrak{K})$, we would expect this to be the already present indexing by $\mathfrak{K}$. Indeed, that $\mathcal{FG}(\mathfrak{K})$ eliminates parameters for this indexing was shown in [37, Lemma 3.4] (see also Corollary VII.56 below).

A similar idea is pursued in the work of Butz and Moerdijk [17] through the use of *enumerated models*.

Definitions VII.54 (§2 [17])**.** Let $\mathbb{T}$ be a geometric theory and let $\mathfrak{K}$ be an infinite set.

(i) A $\mathbb{T}$-model M is said to be $\mathfrak{K}$-*enumerated* if it is $\mathfrak{K}$-indexed and each element $n \in M$ is indexed by infinitely many parameters.

(ii) The *Butz-Moerdijk groupoid* $\mathcal{BM}(\mathfrak{K})$ is the étale-complete groupoid whose objects are all $\mathfrak{K}$-enumerated models.

We will show that both the Forssell groupoids of all $\mathfrak{K}$-indexed models studied in [5], [36], [37] and the Butz-Moerdijk groupoids of all $\mathfrak{K}$-enumerated models of [17] fall within our framework via the following consequence of Theorem VII.8.

Proposition VII.55. *Let $\mathbb{X}$ be an étale complete model groupoid for $\mathbb{T}$ with an indexing by a set of parameters $\mathfrak{K}$ satisfying the following properties.*

(i) The indexing set $\mathfrak{K}$ is infinite.

(ii) The set of models X_0 is closed under finite re-indexing *– by which we mean that for each $M \in X_0$ with an indexing $\mathfrak{K} \twoheadrightarrow M$, then for any injective endomorphism $\mathfrak{K} \rightarrowtail \mathfrak{K}$ whose image is cofinite, X_0 also contains an isomorphic model $M' \cong M$ whose indexing is given by the composite $\mathfrak{K} \rightarrowtail \mathfrak{K} \twoheadrightarrow M \cong M'$, i.e. we can change finitely many of the parameters for any model $M \in X_0$.*

(iii) The set of models X_0 is closed under further indexing *– by which we mean that for each model $M \in X_0$ with an indexing $\mathfrak{K} \twoheadrightarrow M$, and each tuple of parameters $\vec{m}$ not in the domain of $\mathfrak{K} \twoheadrightarrow M$, X_0 also contains the isomorphic model $M' \cong M$ whose indexing is given by any extension of $\mathfrak{K} \twoheadrightarrow M \cong M'$ to include $\vec{m}$ in the domain, i.e. we can add any unused parameters to the indexing of a model $M \in X_0$.*

Then the model groupoid $\mathbb{X}$ eliminates parameters.

Proof. By Remarks VII.7(iii), it suffices to show that, for each tuple of parameters $\vec{m} \in \mathfrak{K}$, $\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}}$ is definable without parameters. We claim that, for each tuple of parameters $\vec{m} \in \mathfrak{K}$,

$$\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}} = \left[\!\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\!\right]_{\mathbb{X}},$$

where the (finite) conjunction $\bigwedge_{m_i=m_j} x_i = x_j$ ranges over the elements $m_i, m_j \in \vec{m}$ that are equal. As observed in Remarks VII.7(v), there is an evident inclusion

$$\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}} \subseteq \left[\!\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\!\right]_{\mathbb{X}},$$

so it remains to demonstrate the reverse inclusion.

Given an element

$$\langle \vec{n}, M \rangle \in \left[\!\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\!\right]_{\mathbb{X}}$$

for which $\vec{m}$ does not appear in the indexing $\mathfrak{K} \rightharpoonup M$, we can use the hypothesis (iii) to deduce the existence of an model $M' \in X_0$ such that $M \cong M'$ and that the tuple $\vec{n}' \in M'$ is indexed by the parameters $\vec{m}$, where $\vec{n}' \in M'$ and $\vec{n} \in M$ are identified under the isomorphism $M \cong M'$. Thus, since $\mathbb{X}$ is étale complete, we obtain that

$$\langle \vec{n}, M \rangle \in \overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}}$$

as required.

It remains to consider the case where the tuple $\vec{m}$ is not disjoint from the domain of the indexing $\mathfrak{K} \rightharpoonup M$ of the model. Since our indexing set is infinite by (i), there exists some injective endomorphism $\mathfrak{K} \rightarrowtail \mathfrak{K}$ whose image is cofinite and does not contain the tuple $\vec{m}$. Thus, by hypothesis (ii), X_0 contains an isomorphic model $M' \cong M$ in whose indexing the tuple of parameters $\vec{m}$ does not appear. Thus, as above we can apply (iii) to deduce that $\vec{n}$ is the image of a tuple of elements indexed by $\vec{m}$ in some model $M'' \in X_0$ under an isomorphism $M'' \cong M' \cong M$, completing the proof. $\square$

Corollary VII.56 (Theorem 1.4.8 [5], Theorem 5.1 [37], [17]). *Let $\mathbb{T}$ be a geometric theory and let $\mathfrak{K}$ be an infinite set.*

(i) The $\mathfrak{K}$-indexed models of $\mathbb{T}$ are conservative if and only if, by endowing $\mathcal{FG}(\mathfrak{K})$ with the logical topologies we obtain a representing groupoid

$$\mathbf{Sh}\left(\mathcal{FG}(\mathfrak{K})^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \simeq \mathcal{E}_{\mathbb{T}}.$$

(ii) The $\mathfrak{K}$-enumerated models of $\mathbb{T}$ are conservative if and only if, by endowing $\mathcal{BM}(\mathfrak{K})$ with the logical topologies we obtain a representing groupoid

$$\mathbf{Sh}\left(\mathcal{BM}(\mathfrak{K})^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \simeq \mathcal{E}_{\mathbb{T}}.$$

Proof. The proof is simply to recognise that Forssell groupoids and Butz-Moerdijk groupoids satisfy the conditions of Proposition VII.55.

We expand the details for Butz-Moerdijk groupoids. By hypothesis, $\mathfrak{K}$ is infinite, and by construction $\mathcal{BM}(\mathfrak{K})$ is closed under further indexing. Since the set of parameters that index each $n \in M \in \mathcal{BM}(\mathfrak{K})$ is infinite, we can change finitely many of the parameters and still end up with a $\mathfrak{K}$-enumerated model, and so $\mathcal{BM}(\mathfrak{K})$ is also closed under finite re-indexing. □

Using Proposition VII.55, we can easily deduce that other similar indexed model groupoids are representing, such as the groupoid of all $\mathfrak{K}$-*finitely indexed* models of a theory $\mathbb{T}$ – i.e. those models that are $\mathfrak{K}$-indexed and whose equivalence class of each $n \in M$ is finite. Also using maximal groupoids, we are able to deduce a useful construction for positing the existence of representing model groupoids with certain structures present in the objects.

Corollary VII.57. *Let $\mathbb{T}$ be a geometric theory and let W be a set of $\mathbb{T}$-models.*

(i) If W is a conservative set of models, then there exists a representing model groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ for $\mathbb{T}$ for which every model $M \in X_0$ is isomorphic to some $M' \in W$.

(ii) If the theory $\mathbb{T}$ has enough points, then there exists a representing model groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ for $\mathbb{T}$ such that X_0 contains W.

Proof. First, suppose that W is a conservative set of models for $\mathbb{T}$. Let $\mathfrak{K}$ be an infinite indexing set for W, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be the étale complete model groupoid of all $\mathfrak{K}$-indexed models of $\mathbb{T}$ that are isomorphic to some model contained in W. By construction, $\mathbb{X}$ is a conservative groupoid, and $\mathbb{X}$ also eliminates parameters since it satisfies the hypotheses of Proposition VII.55.

Now suppose instead that the theory $\mathbb{T}$ has enough points. By [57, Corollary 7.17], we can expand W to a set $W' \supseteq W$ of conservative models for $\mathbb{T}$. We can now apply the above construction to W'. □

VII.5.5 A theory classified by an indexed groupoid

Let $\mathbb{T}$ be a geometric theory over a signature Σ with enough set-based models. The methods of Section VII.5.4 ensure that we can always find a groupoid of Σ-structures, with an indexing by parameters $\mathfrak{K}$, for which the resulting open topological groupoid is a representing groupoid for the theory $\mathbb{T}$.

In this section, we consider the converse problem: given a groupoid $\mathbb{X}$ of Σ-structures with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$, what is a theory classified by the topos $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$ of sheaves on the resulting topological groupoid? It arises that, in general, we cannot choose a theory over the same signature Σ. Instead, we must choose a localic extension. This extends the correspondence between localic extensions and closed subgroups of the permutation group found in [52, Theorem 4.14] and discussed in Section VII.5.3.

Definition VII.58. Let Σ be a signature, and let $\mathbb{X}$ be a groupoid of Σ-structures with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$ (i.e. $\mathbb{X}$ is an indexed model groupoid for $\mathbb{E}_\Sigma$, the empty theory over the signature Σ). We denote by $\Sigma_{\mathfrak{K}\to\mathbb{X}}$ the relational extension of the signature Σ which adds, for each tuple of parameters $\vec{m} \in \mathfrak{K}$, a relation symbol $R_{\vec{m}}$ of the same sort as $\vec{m}$.

The groupoid of Σ-structures $\mathbb{X} = (X_1 \rightrightarrows X_0)$ is automatically a groupoid of $\Sigma_{\mathfrak{K}\to\mathbb{X}}$-structures. For each Σ-structure $M \in X_0$, we interpret $R_{\vec{m}}$ as the subset $\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathbb{X}} \cap M^{\vec{m}}$. The subset

$$[\![\vec{x} : R_{\vec{m}}]\!]_{\mathbb{X}} = \overline{[\![\vec{x} = \vec{m}]\!]}_{\mathbb{X}} \subseteq \coprod_{M \in X_0} M^{\vec{m}}$$

is, by definition, stable, and thus every isomorphism $M \xrightarrow{\alpha} N \in X_1$ preserves the interpretation of the relation $R_{\vec{m}}$. Hence, α is also an isomorphism of $\Sigma_{\mathfrak{K}\to\mathbb{X}}$-structures.

Definition VII.59. Let $\mathbb{X}$ be a groupoid of Σ-structures with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$. We denote by $\mathbb{T}_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$ the *theory of the indexed groupoid*. It is the theory over the signature $\Sigma_{\mathfrak{K}\to\mathbb{X}}$ whose axioms are precisely those sequents $\varphi \vdash_{\vec{x}} \psi$ over $\Sigma_{\mathfrak{K}\to\mathbb{X}}$ which are satisfied in all structures $M \in X_0$ (once each $M \in X_0$ is interpreted as a $\Sigma_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$-structure).

Corollary VII.60. *Let $\mathfrak{K} \twoheadrightarrow \mathbb{X}$ be an indexed groupoid of Σ-structures. There is an equivalence of topoi*

$$\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right) \simeq \mathcal{E}_{\mathbb{T}_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}}.$$

Proof. By the definition of the theory $\mathbb{T}_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$, the groupoid $\mathbb{X}$ is a conservative groupoid for $\mathbb{T}_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$. Next, by the construction of the signature $\Sigma_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$, the groupoid $\mathbb{X}$ eliminates parameters as a groupoid of $\Sigma_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$-structures. Explicitly, for each tuple of parameters, we have that

$$\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathbb{X}} = [\![\vec{x} : R_{\vec{m}}]\!]_{\mathbb{X}}.$$

Thus, by Theorem VII.8, the topos $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right)$ classifies the theory $\mathbb{T}_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$. □

Example VII.61 (The theory of a generic Dedekind section)**.** Let $\mathbb{X}$ be a groupoid of Σ-structures with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$. While the signature $\Sigma_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$ constructed in Definition VII.58 ensures that $\mathbb{X}$ eliminates parameters over the signature $\Sigma_{\mathfrak{K}\twoheadrightarrow\mathbb{X}}$, we may however wish to refrain from adding too many symbols to our signature.

Evidently, we do not need to add a new relation symbol $R_{\vec{m}}$ for every tuple of parameters $\vec{m} \in \mathfrak{K}$, but only those for which the orbit $\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathbb{X}}$ is not definable without parameters. By making astute choices about how to expand the signature, we can minimise the number of new symbols we must add.

Recall from Example VII.40 that $\mathbb{R} + \mathbb{R}$ is a model for the theory $\mathbb{DLO}_\infty$ that is not ultrahomogeneous, and consequently the automorphism group $\mathrm{Aut}(\mathbb{R} + \mathbb{R})$ does not eliminate parameters. We describe a theory classified by the topos $\mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R})$ using the above techniques.

For $i = 1, 2$, the automorphism group $\mathrm{Aut}(\mathbb{R} + \mathbb{R})$ acts transitively on the subset $\{ i \} \times \mathbb{R} \subseteq \mathbb{R} + \mathbb{R}$, i.e. for any $r \in \mathbb{R}$,

$$\overline{[\![x = (i, r)]\!]}_{\mathrm{Aut}(\mathbb{R}+\mathbb{R})} = \{ i \} \times \mathbb{R} \subseteq \mathbb{R} + \mathbb{R},$$

and so we are motivated to consider the localic extension of the theory of dense linear orders without endpoints by the addition of a pair of unary relation symbols U_1 and U_2, where these symbols are interpreted in the model $\mathbb{R} + \mathbb{R}$ as the subsets

$$[\![U_1(x)]\!]_{\mathbb{R}+\mathbb{R}} = \{ 1 \} \times \mathbb{R},$$
$$\text{and } [\![U_2(x)]\!]_{\mathbb{R}+\mathbb{R}} = \{ 2 \} \times \mathbb{R}.$$

The automorphism group $\mathrm{Aut}(\overline{\mathbb{R}+\mathbb{R}})$ eliminates parameters over this expanded signature. Namely, we have that $[\![\vec{x} = \vec{m}]\!]_{\mathrm{Aut}(\mathbb{R}+\mathbb{R})}$ is given by

$$\left[\!\!\left[\vec{x} : \bigwedge_{\substack{m_i,m_j\in\vec{m},\\ m_i=m_j}} x_i = x_j \wedge \bigwedge_{\substack{m_i,m_j\in\vec{m},\\ m_i<m_j}} x_i < x_j \wedge \bigwedge_{\substack{m_i\in\vec{m},\\ m_i\in\{1\}\times\mathbb{R}}} U(x_i) \wedge \bigwedge_{\substack{m_j\in\vec{m},\\ m_j\in\{2\}\times\mathbb{R}}} U(x_j) \right]\!\!\right]_{\mathrm{Aut}(\mathbb{R}+\mathbb{R})}.$$

Thus, in a manner similar to Corollary VII.60, we deduce that the topos $\mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R})$ classifies the localic expansion of $\mathbb{DLO}_\infty$ by two unary predicates whose axioms are those sequents satisfied in the model $\mathbb{R}+\mathbb{R}$, which we denote by $\mathbb{T}_{\mathbb{R}+\mathbb{R}}$. The sequents

$$\begin{array}{ll}
U_1(x) \wedge U_2(x) \vdash_x \bot, & x < y \vdash_{x,y} U_1(x) \vee U_2(y), \\
\top \vdash_\emptyset \exists x\, U_1(x), & \top \vdash_\emptyset \exists y\, U_2(y), \\
U_1(x) \wedge y < x \vdash_{x,y} U_1(y), & U_2(y) \wedge y < x \vdash_{x,y} U_2(y), \\
U_1(x) \vdash_x \exists y\, U_1(y) \wedge x < y, & U_2(y) \vdash_y \exists x\, U_2(x) \wedge x < y
\end{array}$$

(in addition to those for a dense linear order without endpoints) suffice to generate this theory.

The theory $\mathbb{T}_{\mathbb{R}+\mathbb{R}}$ can be likened to a *theory of Dedekind sections* for an arbitrary dense linear order without endpoints[1]. Indeed, the rationals $\mathbb{Q}$ can be made into a model of $\mathbb{T}_{\mathbb{R}+\mathbb{R}}$ with the interpretations

$$[\![U_1(x)]\!]_{\mathbb{Q}} = (-\infty, a) \text{ and } [\![U_2(x)]\!]_{\mathbb{Q}} = (a, \infty)$$

for any irrational a. Of course, not every automorphism of $\mathbb{Q}$ as a linear order will preserve the further U_1 and U_2 structure. In contrast, $\mathbb{R}$ does not admit an interpretation as a $\mathbb{T}_{\mathbb{R}+\mathbb{R}}$-model.

VII.6 The theory of algebraic integers

We have seen in Section VII.5.1 that the examples of representing groups and groupoids considered in [11] and [21] can be subsumed by the classification result Theorem VII.8. Similarly, in Section VII.5.4 we showed that the 'maximal' representing groupoids constructed in [5], [17], [36], [37] also fall within the scope of Theorem VII.8. Examples of representing groupoids that do not directly originate via the methods exposited in the surrounding literature have been given in Examples VII.47(ii) and Example VII.49.

As a worked example, in this section we study in further detail another representing groupoid that does not arise from the previous approaches found in the literature. The theory we consider is the theory of *algebraic integers*.

[1]In [123, §3.5], Vickers describes, as a localic expansion of the theory of dense linear orders without endpoints, a theory of Dedekind sections on the rationals. Tacitly, an interpretation of the rationals is fixed by introducing a constant symbol c_q for each rational $q \in \mathbb{Q}$ and axioms

(i) $\top \vdash c_p < c_q$, for each pair of rationals p, q with $p < q$,

(ii) and $\top \vdash_x \bigvee_{q\in\mathbb{Q}} x = c_q$.

Consequently, there are no non-trivial isomorphisms of models.

The theory of algebraic integers. For each prime p, the monic minimal polynomial of each element of the field $\overline{\mathbb{Z}/\langle p\rangle}$ has integer coefficients. In this sense, the algebraic numbers and algebraic integers modulo p coincide, and the field $\overline{\mathbb{Z}/\langle p\rangle}$ is a model for a theory of algebraic integers. Along with the standard ring of algebraic integers $\overline{\mathbb{Z}}$, these rings can be axiomatised as follows.

Definition VII.62. We denote by $\mathbb{AI}$ the (geometric) theory of *algebraic integers*. It is the single-sorted theory over the signature of rings whose axioms are the following:

(i) the standard axioms of a commutative ring,

(ii) the sequent
$$x \cdot y = 0 \vdash_{x,y} x = 0 \vee y = 0,$$
expressing that any model is an integral domain,

(iii) the sequent
$$\top \vdash_x \bigvee_{q \in \mathbb{Z}[x]_{\mathrm{monic}}} q(x),$$
where $\mathbb{Z}[x]_{\mathrm{monic}}$ denotes the set of *monic* polynomials with integer coefficients, expressing that every element is in the *integral closure* of the prime subring,

(iv) and for each $n \in \mathbb{N}$, the sequent
$$\top \vdash_{x_{n-1},\dots,x_0} \exists y \; y^n + x_{n-1}y^{n-1} + \cdots + x_0,$$
expressing that the model is algebraically closed with respect to monic polynomials.

Lemma VII.63. *Every model of $\mathbb{AI}$ is either isomorphic to $\overline{\mathbb{Z}}$ or to $\overline{\mathbb{Z}/\langle p\rangle}$ for some prime p.*

Proof. A model R of $\mathbb{AI}$ is an integral domain, and it is isomorphic to $\overline{\mathbb{Z}}$ or $\overline{\mathbb{Z}/\langle p\rangle}$ for some prime p depending on the characteristic of R. We will show that if R has characteristic 0, then $R \cong \overline{\mathbb{Z}}$. The proof in the case where R has finite characteristic is almost identical.

Since R is an integral domain, we can form its field of fractions $\mathrm{Frac}(R)$ as well as the algebraic closure $\overline{\mathrm{Frac}(R)}$ of this field. Subsequently, there exist isomorphic copies of $\mathbb{Z}, \overline{\mathbb{Z}}, \mathbb{Q}$ and $\overline{\mathbb{Q}}$ inside $\overline{\mathrm{Frac}(R)}$ along with the inclusions of rings

$$\begin{tikzcd}
 & R \arrow[rr, hook] & & \mathrm{Frac}(R) \arrow[dr, hook] & \\
\mathbb{Z} \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, hook] & & \mathbb{Q} \arrow[ur, hook] \arrow[dr, hook] & & \overline{\mathrm{Frac}(R)}. \\
 & \overline{\mathbb{Z}} \arrow[rr, hook] & & \overline{\mathbb{Q}} \arrow[ur, hook] &
\end{tikzcd}$$

Viewed as subrings of $\overline{\mathrm{Frac}(R)}$, the condition that R is algebraic over its prime subring ensures that $R \subseteq \overline{\mathbb{Z}}$, while the condition that R is algebraically closed with respect to monic polynomials ensures the converse inclusion. □

Since $\mathbb{AI}$ is not an atomic theory (for example, there is no single geometric formula that is provably equivalent to the infinite conjunction $\bigwedge_{p \text{ prime}} p \neq 0$), the classifying topos $\mathcal{E}_{\mathbb{AI}}$ cannot be equivalent to a topos of sheaves on a simple disjoint coproduct of automorphism groups as in Corollary VII.42. Instead, we must search further afield for a representing groupoid.

We note as well that the standard ring of algebraic integers $\overline{\mathbb{Z}}$ plays a special role amongst all models of the theory $\mathbb{AI}$. It is not a conservative model – this is clear since $\overline{\mathbb{Z}}$ satisfies the sequent

$$\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0 \vdash \bot,$$

but $\mathbb{AI}$ has a model of cardinality p. However, $\overline{\mathbb{Z}}$ does have the property that $\overline{\mathbb{Z}}$ satisfies a (geometric) sentence φ if and only if $\mathbb{T}$ proves the sequent $\top \vdash \varphi$. For this reason, we will say that $\overline{\mathbb{Z}}$ is *sentence-complete*. Below, we construct a representing groupoid for $\mathbb{AI}$ where the fact that $\overline{\mathbb{Z}}$ is a sentence-complete model can be captured topologically, as seen in Corollary VII.67.

A representing groupoid for the theory of algebraic integers. Let $a \in \overline{\mathbb{Z}}$ be an algebraic integer, and let q_a be its minimal polynomial. For any ring homomorphism $f : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}/\langle p \rangle}$, the image $f(a)$ is also a root of the polynomial q_a. However, over $\mathbb{Z}/\langle p \rangle$, the polynomial q_a may no longer be irreducible. Suppose that q_a factors into irreducible polynomials as $q_a^1 q_a^2 \dots q_a^k$ over $\mathbb{Z}/\langle p \rangle$. Then $f(a) \in \overline{\mathbb{Z}/\langle p \rangle}$ has as its minimal polynomial q_a^i for some i.

Assuming the axiom of choice, there exists a maximal ideal M of $\overline{\mathbb{Z}}$ which contains both the prime p and $q_a^i(a)$. Taking the quotient ring yields a field $\overline{\mathbb{Z}}/M$ which is moreover an algebraic closure for its prime subfield $\mathbb{Z}/\langle p \rangle$. Hence, by the uniqueness of algebraic closures, we deduce the existence of a surjective ring homomorphism

$$\pi_M : \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{Z}}/M \cong \overline{\mathbb{Z}/\langle p \rangle}$$

with the property that $\pi_M(a)$ has minimal polynomial q_a^i.

Definition VII.64. Let $\mathcal{AI} = (\mathcal{AI}_1 \rightrightarrows \mathcal{AI}_0)$ denote the étale complete model groupoid for $\mathbb{AI}$ whose underlying set of objects $\mathcal{AI}_0$ is constructed as follows:

(i) $\mathcal{AI}_0$ contains one copy of the model $\overline{\mathbb{Z}}$;

(ii) we add, for each prime p and each maximal ideal $M \subseteq \overline{\mathbb{Z}}$ containing p, a copy of the model $\overline{\mathbb{Z}/\langle p \rangle}$.

That is, $\mathcal{AI}$ is the groupoid

$$\mathrm{Aut}\left(\overline{\mathbb{Z}}\right) + \coprod_{p \text{ prime}} \mathbf{ConGrpd}\left(\left\{ \begin{matrix} M \subseteq \overline{\mathbb{Z}} \text{ a maximal} \\ \text{ideal containing } p \end{matrix} \right\}, \mathrm{Aut}\left(\overline{\mathbb{Z}/\langle p \rangle}\right)\right),$$

where we use the notation $\mathbf{ConGrpd}(Y, G)$, for a set Y and a group G, to denote the (unique) connected groupoid whose objects are Y and $\mathbf{ConGrpd}(Y, G)(y, y') = G$, for each pair $y, y' \in Y$.

We endow $\mathcal{AI}$ with the indexing whose set of parameters are the algebraic integers $\overline{\mathbb{Z}}$. The model $\overline{\mathbb{Z}}$ is given the trivial indexing of itself by its own elements. Meanwhile, the indexing of the model $\overline{\mathbb{Z}/\langle p\rangle}$ corresponding to the maximal ideal $M \subseteq \overline{\mathbb{Z}}$ is determined by the surjective frame homomorphism

$$\pi_M \colon \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{Z}}/M \cong \overline{\mathbb{Z}/\langle p\rangle}$$

To make this indexing explicit, we will not abuse notation (as we have done in the rest of this chapter) and instead denote the interpretation of the parameter $a \in \overline{\mathbb{Z}}$ in a copy of $\overline{\mathbb{Z}/\langle p\rangle}$ by $\pi_M(a)$.

Proposition VII.65. *The groupoid $\mathcal{AI}$ of $\mathbb{AI}$-models, with the indicated indexing $\overline{\mathbb{Z}} \twoheadrightarrow \mathcal{AI}$, is conservative and eliminates parameters. Therefore, there is an equivalence of topoi*

$$\mathcal{E}_{\mathbb{AI}} \simeq \mathbf{Sh}\left(\mathcal{AI}^{\tau\text{-log}_1}_{\tau\text{-log}_0}\right).$$

Proof. By Lemma VII.63, the set of objects $\mathcal{AI}_0$ is a conservative set of models. By Remarks VII.7(iii), to show that $\mathcal{AI}$ eliminates parameters it suffices to demonstrate that, for each tuple of parameters $\vec{a} \in \overline{\mathbb{Z}}$, the orbit $\overline{[\![\, \vec{x} = \vec{a}\,]\!]}_{\mathcal{AI}}$ is definable without parameters. We claim that

$$\overline{[\![\, \vec{x} = \vec{a}\,]\!]_{\mathcal{AI}}} = [\![\, \vec{x} : q_{a_1}(x_1) = 0 \wedge q_{a_2}(x_1, x_2) = 0 \wedge \cdots \wedge q_{a_n}(x_1, \ldots, x_n) = 0 \,]\!]_{\mathcal{AI}},$$

where $q_{a_i}(a_1, \ldots, a_{i-1}, x_i)$ is a minimal polynomial of $a_i \in \vec{a}$ over the ring extension $\mathbb{Z}[a_1, \ldots, a_{i-1}]$.

By definition, the tuple $\vec{a} \in \overline{\mathbb{Z}}$ satisfies the formula

$$\overline{\mathbb{Z}} \vDash q_{a_1}(a_1) = 0 \wedge q_{a_2}(a_1, a_2) = 0 \wedge \cdots \wedge q_{a_n}(a_1, \ldots, a_n) = 0.$$

Moreover, since the interpretation of the parameters $\vec{a}$ in any other model $\overline{\mathbb{Z}/\langle p\rangle} \in \mathcal{AI}_0$ is determined by a ring homomorphism $\pi_M \colon \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{Z}/\langle p\rangle}$, we conclude that

$$\begin{aligned} \overline{\mathbb{Z}/\langle p\rangle} &\vDash \pi_M(q_{a_1}(a_1)) = 0 \wedge \cdots \wedge \pi_M(q_{a_n}(a_1, \ldots, a_n)) = 0, \\ &\vDash q_{a_1}(\pi_M(a_1)) = 0 \wedge \cdots \wedge q_{a_n}(\pi_M(a_1), \ldots, \pi_M(a_n)) = 0. \end{aligned}$$

Consequently, there is an inclusion

$$[\![\, \vec{x} = \vec{a}\,]\!]_{\mathcal{AI}} \subseteq \overline{[\![\, \vec{x} = \vec{a}\,]\!]_{\mathcal{AI}}} \subseteq [\![\, \vec{x} : q_{a_1}(x_1) = 0 \wedge \cdots \wedge q_{a_n}(x_1, \ldots, x_n) = 0 \,]\!]_{\mathcal{AI}}.$$

We turn to the converse inclusion. Since the automorphism group $\mathrm{Aut}\left(\overline{\mathbb{Z}}\right)$ acts transitively on the set of solutions to an irreducible polynomial, we have that

$$\begin{aligned} [\![\, \vec{x} : q_{a_1}(x_1) = 0 \wedge \cdots \wedge q_{a_n}(x_1, \ldots, x_n) = 0 \,]\!]_{\mathcal{AI}} \cap \overline{\mathbb{Z}} &\subseteq \overline{[\![\, \vec{x} = \vec{a}\,]\!]}_{\mathrm{Aut}(\overline{\mathbb{Z}})}, \\ &= \overline{[\![\, \vec{x} = \vec{a}\,]\!]}_{\mathcal{AI}} \cap \overline{\mathbb{Z}}. \end{aligned}$$

Now let $\vec{w} \in \overline{\mathbb{Z}/\langle p\rangle}$ be a tuple for which

$$\overline{\mathbb{Z}/\langle p\rangle} \vDash q_{a_1}(w_1) = 0 \wedge q_{a_2}(w_1, w_2) = 0 \wedge \cdots \wedge q_{a_n}(w_1, \ldots, w_n) = 0,$$

and let $q_{w_i}(x_1, \ldots, x_i)$ be a minimal polynomial of w_i over $\mathbb{Z}/\langle p\rangle[w_1, \ldots, w_{i-1}]$. By Zorn's lemma, we can extend the non-trivial ideal on $\overline{\mathbb{Z}}$ generated by the set

$$\{\, p, q_{w_1}(a_1), \ldots, q_{w_n}(a_1, \ldots, a_n)\,\}$$

to a maximal ideal $M \subseteq \overline{\mathbb{Z}}$. Hence, there is an isomorphic copy of $\overline{Z}/M \cong \overline{\mathbb{Z}/\langle p\rangle}$ in $\mathcal{AI}_0$ in which $\pi_M(a_i)$ is a root of the irreducible polynomial $q_{w_i}(\pi_M(a_1), \ldots, \pi_M(a_{i-1}), x)$. Thus, there exists an isomorphism $\overline{Z}/M \cong \overline{\mathbb{Z}/\langle p\rangle}$ sending $\pi_M(\vec{a})$ to $\vec{w}$, completing the proof of the converse inclusion

$$[\![\, \vec{x} : q_{a_1}(x_1) = 0 \wedge \cdots \wedge q_{a_n}(x_1, \ldots, x_n) = 0 \,]\!]_{\mathcal{AI}} \subseteq \overline{[\![\, \vec{x} = \vec{a} \,]\!]}_{\mathcal{AI}}.$$

□

Properties of the space of objects. We now describe the space of objects $\mathcal{AI}_0^{\tau\text{-log}_0}$ in more detail. Eventually, we will observe that $\mathcal{AI}_0^{\tau\text{-log}_0}$ can be described, up to homeomorphism, in entirely topological terms. Recall that a basic open of $\mathcal{AI}_0^{\tau\text{-log}_0}$ is given by the interpretation of a sentence with parameters $[\![\, \vec{m} : \varphi \,]\!]_{\mathcal{AI}} \subseteq \mathcal{AI}_0$. Recall also from Remark VII.15 that it suffices to consider only the atomic sentences with parameters, which in the case of rings amounts to formulae of the form $q(\vec{a}) = 0$, for some tuple of parameters $\vec{a}$ and a polynomial q.

We first note that the subset of $\mathcal{AI}_0$ consisting only of models of the form $\overline{\mathbb{Z}/\langle p\rangle}$ is an open subset in τ-$\log_0$. Namely, it is the subset

$$[\![\, p = 0 \,]\!]_{\mathcal{AI}} \cong \left\{ M \subseteq \overline{\mathbb{Z}} \;\middle|\; p \in M,\ M \text{ a maximal ideal} \right\} \subseteq \mathcal{AI}_0.$$

Lemma VII.66. *For each prime p, the subspace*

$$[\![\, p = 0 \,]\!]_{\mathcal{AI}} \cong \left\{ M \subseteq \overline{\mathbb{Z}} \;\middle|\; p \in M,\ M \textit{ a maximal ideal} \right\}$$

of $\mathcal{AI}_0^{\tau\text{-log}_0}$ is homeomorphic to the Cantor space $2^{\mathbb{N}}$.

Proof. We first remark that the Cantor space $2^{\mathbb{N}}$ is homeomorphic to any uncountable closed subspace of itself (this is a consequence of Brouwer's characterisation of the Cantor space). Thus, it suffices to show that $[\![\, p = 0 \,]\!]_{\mathcal{AI}}$, an uncountable space, is homeomorphic to a closed subspace of $2^{\mathbb{N}}$.

There is an evident inclusion of sets

$$[\![\, p = 0 \,]\!]_{\mathcal{AI}} \cong \left\{ M \subseteq \overline{\mathbb{Z}} \;\middle|\; p \in M,\ M \text{ a maximal ideal} \right\} \subseteq 2^{\overline{\mathbb{Z}}} \cong 2^{\mathbb{N}}.$$

We must first show that the induced topology on $[\![\, p = 0 \,]\!]_{\mathcal{AI}}$ as a subspace of $\mathcal{AI}_0^{\tau\text{-log}_0}$ is the same as that induced as a subspace of $2^{\overline{\mathbb{Z}}}$. The topology on $2^{\overline{\mathbb{Z}}}$ is generated by the basic opens $\{\, M \mid a \in M \,\}$ and $\{\, M \mid a \notin M \,\}$, for each algebraic integer $a \in \overline{\mathbb{Z}}$.

Under the bijection $[\![\, p = 0 \,]\!]_{\mathcal{AI}} \cong \{\, M \mid p \in M \,\}$, the subset $\{\, M \mid a \in M \,\}$ corresponds to the open

$$[\![\, a = 0 \,]\!]_{\mathcal{AI}} \cap [\![\, p = 0 \,]\!]_{\mathcal{AI}}.$$

To show that $\{M \mid a \notin M\}$ is also open, we note that if $a \notin M$, then $\pi_M(a)$ is a non-zero element of the field $\overline{\mathbb{Z}}/M \cong \overline{\mathbb{Z}/\langle p\rangle}$ and hence invertible. Thus, under the bijection $[\![p = 0]\!]_{\mathcal{AI}} \cong \{M \,|\, p \in M\}$, we have that

$$[\![\exists y\; a \cdot y = 1]\!]_{\mathcal{AI}} \cap [\![p = 0]\!]_{\mathcal{AI}} \cong \{M \mid a \notin a\}.$$

Hence, the topology on $[\![p = 0]\!]_{\mathcal{AI}}$ induced as a subspace of $\mathcal{AI}_0^{\tau\text{-}\log_0}$ contains that as induced as a subspace of $2^{\overline{\mathbb{Z}}}$. For the reverse inclusion of topologies, by Remark VII.15 it suffices to note that

$$[\![q(\vec{a}) = 0]\!]_{\mathcal{AI}} \cap [\![p = 0]\!]_{\mathcal{AI}} \cong \{M \,|\, q(\vec{a}) \in M\}.$$

Thus, $[\![p = 0]\!]_{\mathcal{AI}}$ is a subspace of $2^{\overline{\mathbb{Z}}}$.

It remains to show that $[\![p = 0]\!]_{\mathcal{AI}}$ is a closed subset of $2^{\overline{\mathbb{Z}}}$. It is straightforward to demonstrate that the complement $2^{\overline{\mathbb{Z}}} \setminus [\![p = 0]\!]_{\mathcal{AI}}$ is open once we recall that, in $\overline{\mathbb{Z}}$, the maximal ideals are precisely the non-zero prime ideals. If P fails any of the conditions to be a prime ideal containing p, it easy to find an open neighbourhood of P that is contained entirely in the complement $2^{\overline{\mathbb{Z}}} \setminus [\![p = 0]\!]_{\mathcal{AI}}$. As an example, if $P \subseteq \overline{\mathbb{Z}}$ contains the product $a \cdot b$ but neither a nor b, i.e. P is not *prime*, then

$$\left\{P \subseteq \overline{\mathbb{Z}} \,\middle|\, a \cdot b \in P\right\} \cap \left\{P \subseteq \overline{\mathbb{Z}} \,\middle|\, a \notin P\right\} \cap \left\{P \subseteq \overline{\mathbb{Z}} \,\middle|\, b \notin P\right\}$$

is such an open neighbourhood of P. Thus, $[\![p = 0]\!]_{\mathcal{AI}} \subseteq 2^{\overline{\mathbb{Z}}}$ is closed, from which the result follows. □

Next, we deduce that the point $\overline{\mathbb{Z}} \in \mathcal{AI}_0$ is a *universal accumulation point* of $\mathcal{AI}_0$, by which we mean that $\overline{\mathbb{Z}}$ is an accumulation point of every subset $S \subseteq \mathcal{AI}_0 \setminus \{\overline{\mathbb{Z}}\}$, or rather: the only open containing $\overline{\mathbb{Z}}$ is the whole space. This is because if $\overline{\mathbb{Z}}$ is contained in a basic open $[\![q(\vec{a}) = 0]\!]_{\mathcal{AI}}$, i.e. $\overline{\mathbb{Z}} \vDash q(\vec{a}) = 0$, then $\overline{\mathbb{Z}/\langle p\rangle} \vDash q(\pi_M(\vec{a}))$ for each maximal ideal M of $\overline{\mathbb{Z}}$ containing p. Hence, each copy of $\overline{\mathbb{Z}/\langle p\rangle} \in \mathcal{AI}_0$ is also contained in the open $[\![q(\vec{a}) = 0]\!]_{\mathcal{AI}}$, and thus $[\![q(\vec{a}) = 0]\!]_{\mathcal{AI}} = \mathcal{AI}_0$. As a consequence, we deduce that:

Corollary VII.67. *The algebraic integers $\overline{\mathbb{Z}}$ are a 'sentence-complete' model of the theory $\mathbb{AI}$.*

Combining the above, we are able to give an entirely topological description of the space $\mathcal{AI}_0^{\tau\text{-}\log_0}$ devoid of any mention of algebraic structures from which it derives.

Corollary VII.68. *The space of objects $\mathcal{AI}_0^{\tau\text{-}\log_0}$ is obtained by the addition of a universal accumulation point to the coproduct of countably many copies of the Cantor space.*

VII.7 Representing groupoids for doctrines

While conservative sets of models are commonly considered in other fragments of logic, we have defined elimination of parameters only for model groupoids of geometric theories. The reader may rightly wonder how our theory of elimination of parameters generalises beyond geometric logic, and thus how to apply our classification of representing open topological groupoids to theories of from other fragments of logic, e.g. classical logic.

Recall that, irrespective of the underlying syntax, the classifying topos of a theory (or indeed a doctrinal site) also classifies a geometric theory to which we can apply the classification result of Theorem VII.8. The purpose of Part A was to identify a suitable choice of such a geometric theory/doctrine – the geometric completion. The properties of conservativity and elimination of parameters identified in Theorem VII.8, defined on the geometric completion, can be translated back into properties on the original theory (respectively, doctrine).

This section contains that calculation. Hence, we will obtain a classification of the representing open topological groupoids of any formal system of predicate reasoning, as represented by a doctrinal site. We outline the necessary adjustments that must be made to the method pursued in Sections VII.2 to VII.4 (since we no longer assume that our geometric doctrine is fibred over a category of contexts $\mathbf{Con}_N$) before giving the statement of the classification result for doctrinal sites in Theorem VII.72.

Indexed models of a doctrine. Recall that a (set-based) model M of a doctrinal site (P, J) is a morphism of doctrinal sites $(P, J) \to (\mathscr{P}, K_{\mathscr{P}})$, which we suggestively write as being composed of the pair consisting of

(i) a flat functor $M \colon C \to \mathbf{Sets}$,

(ii) and a pseudo-natural transformation $[\![-]\!]_M \colon P \Rightarrow \mathscr{P} \circ M^{\mathrm{op}}$ for which the induced functor

$$M \rtimes [\![-]\!]_M \colon (C \rtimes P, J) \longrightarrow (\mathbf{Sets} \rtimes \mathscr{P}, K_{\mathscr{P}})$$

is a morphism of sites.

By the universal property of the geometric completion, the pseudo-natural transformation $[\![-]\!]_M$ can be uniquely extended to a natural transformation

$$\begin{tikzcd} P \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "{[\![-]\!]_M}"'] & \mathfrak{Z}(P, J) \arrow[d, dashed] \\ & \mathscr{P} \end{tikzcd}$$

that yields a morphism of geometric doctrines $\mathfrak{Z}(P, J) \to \mathscr{P}$. We will not differentiate between the pseudo-natural transformation $[\![-]\!]_M \colon P \Rightarrow \mathscr{P} \circ M^{\mathrm{op}}$ and its extension to the geometric completion $\mathfrak{Z}(P, J)$.

Just as in Definition VII.1, we can define a notion of indexing for the models of a doctrinal site.

Definition VII.69. Let M be a model of the doctrinal site (P, J). An *indexing* of M consists of

(i) a covariant presheaf of parameters $\mathfrak{K} \colon C \to \mathbf{Sets}$,

(ii) and, for each object $c \in C$, a partial surjection $\mathfrak{R}_c \twoheadrightarrow M(c)$ which is natural in the sense that the square

$$\begin{tikzcd} \mathfrak{R}_c \arrow[r, "f^{\mathfrak{R}}"] \arrow[d, two heads] & \mathfrak{R}_d \arrow[d, two heads] \\ M(c) \arrow[r, "M(f)"] & M(d). \end{tikzcd} \tag{VII.iii}$$

commutes.

We write $\mathfrak{R} \twoheadrightarrow M$ to denote such an indexing.

Remark VII.70. Let $\mathbb{T}$ be a geometric theory. Recall that a model M of $\mathbb{T}$ in the usual sense is equivalent to a model of the associated geometric doctrine $F^{\mathbb{T}}$. The two notions of indexings of M given in Definition VII.1 and Definition VII.69 are not identical, but can be identified up to equivalence.

Suppose that, for each singleton variable x of the theory $\mathbb{T}$, there is a given partial surjection $\mathfrak{R}_x \twoheadrightarrow M^x$, i.e. M is an indexed model according to Definition VII.1. We obtain a presheaf of parameters $\mathfrak{R}\colon \mathbf{Con}_N^{\mathrm{op}} \to \mathbf{Sets}$ by setting $\mathfrak{R}_{\vec{x}}$ as the product $\prod_{x_i \in \vec{x}} \mathfrak{R}_{x_i}$ and, for each relabelling of contexts $\sigma\colon \vec{y} \to \vec{x}$, setting $\sigma^{\mathfrak{R}}$ as the universally induced map

$$\begin{tikzcd} \mathfrak{R}_{\vec{x}} \arrow[r, dashed, "\sigma^{\mathfrak{R}}"] \arrow[d, "\mathrm{pr}_{\sigma(\vec{x})}"] & \prod_{y_i \in \vec{y}} \mathfrak{R}_{y_i} = \mathfrak{R}_{\vec{y}} \arrow[d, "\mathrm{pr}_{y_i}"] \\ \mathfrak{R}_{\sigma(y_i)} \arrow[r, equal] & \mathfrak{R}_{y_i}. \end{tikzcd}$$

By a similar universal construction, there exists a natural partial surjection $\mathfrak{R}_{\vec{x}} \twoheadrightarrow M^{\vec{x}}$ for each context $\vec{x}$.

Conversely, an indexing $\mathfrak{R} \twoheadrightarrow M$ of M according to Definition VII.69 evidently yields an indexing of M in the sense of Definition VII.1 since, for each singleton context x, we have a partial surjection $\mathfrak{R}_x \twoheadrightarrow M^x$.

In fact, these two processes are mutually inverse up to equivalence. The only discrepancy arises because, for an arbitrary presheaf of parameters $\mathfrak{R}\colon \mathbf{Con}_N^{\mathrm{op}} \to \mathbf{Sets}$, it is not necessarily the case that $\mathfrak{R}_{\vec{x}} \cong \prod_{x_i \in \vec{x}} \mathfrak{R}_{x_i}$. However, the induced indexings on the model M are equivalent since the partial surjection $\mathfrak{R}_{\vec{x}} \twoheadrightarrow M^{\vec{x}}$ necessarily factors as

$$\begin{tikzcd} \mathfrak{R}_{\vec{x}} \arrow[d, dashed, two heads] \arrow[dr, bend left] \arrow[dd, bend right=60, two heads] & \\ \prod_{x_i \in \vec{x}} \mathfrak{R}_{x_i} \arrow[r] \arrow[d, two heads] & \mathfrak{R}_{x_i} \arrow[d, two heads] \\ M^{\vec{x}} \arrow[r] & M^{x_i}. \end{tikzcd}$$

Classification of the representing open topological groupoids of a doctrine. In the aid of intuition, during Sections VII.2 to VII.4 we worked in the familiar language of a theory of geometric logic, or less transparently, by Proposition III.42, an internal locale of a topos of the form $\mathbf{Sets}^{\mathbf{Con}_N}$. However, every step of the proof followed in Sections VII.2 to VII.4 generalises readily to doctrines over an arbitrary base category.

The only results whose generalisations to arbitrary doctrines deserve clarification are Lemma VII.16 and Proposition VII.17.

Proposition VII.71. *Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{PreOrd}$ be a doctrine with a classifying topos $\mathcal{E}_P$ and let X_0 be a set of models of P. A topology τ_0 on X_0 is a factoring topology, by which we mean that there is a factorisation of the canonical geometric morphism*

$$\mathbf{Sets}^{X_0} \xrightarrow{j} \mathbf{Sh}(X_0^{\tau_0}) \dashrightarrow \mathcal{E}_P,$$

if and only if there is an indexing of each $M \in X_0$ by a presheaf of parameters $\mathfrak{K}\colon \mathcal{C} \to \mathbf{Sets}$ such that τ_0 contains the corresponding logical topology, i.e. the topology generated by the basic opens

$$[\![\, m : U \,]\!]_{X_0} = \{\, M \in X_0 \,|\, m \in [\![\, U \,]\!]_M \subseteq M(c) \,\}$$

for each $c \in \mathcal{C}$, $m \in \mathfrak{K}_c$ and $U \in P(c)$.

Proof. By an evident generalisation of Proposition VII.12, a topology τ_0 is a factoring topology if and only if there is a factorisation

$$\begin{tikzcd} \mathbf{Sets}^{X_0} & \\ \mathbf{Sh}(X_0^{\tau_0}) \arrow[u, "j"] & \mathcal{C} \rtimes P, \arrow[l, dashed] \arrow[ul, "{[\![-]\!]_{X_0}}"'] \end{tikzcd}$$

if and only if, for each object $c \in \mathcal{C}$, there is a topology T_c on $\coprod_{M \in X_0} M(c)$ such that

(i) the projection

$$\pi_c\colon \left(\textstyle\coprod_{M \in X_0} M(c)\right)^{T_c} \longrightarrow X_0^{\tau_0}$$

is a local homeomorphism,

(ii) for each $U \in P(c)$, the subset

$$[\![\, U \,]\!]_{X_0} = \coprod_{M \in X_0} [\![\, U \,]\!]_M \subseteq \coprod_{M \in X_0} M(c)$$

is open in the topology T_c,

(iii) and for each arrow $d \xrightarrow{f} c \in \mathcal{C}$, the map

$$\left(\textstyle\coprod_{M \in X_0} M(d)\right)^{T_d} \xrightarrow{f_{X_0} = \coprod_{M \in X_0} M(f)} \left(\textstyle\coprod_{M \in X_0} M(c)\right)^{T_c}$$

is continuous.

First, suppose that $M \in X_0$ is indexed by a presheaf of parameters $\mathfrak{K}: \mathcal{C} \to \mathbf{Sets}$. We identify a topology T_c on $\coprod_{M\in X_0} M(c)$ which satisfies conditions (i) to (iii) when X_0 is endowed with the topology τ-$\log_0$ generated by the basic opens $[\![\, m : U \,]\!]_{X_0}$.

The topology T_c we choose is the obvious generalisation of the topology generated by definables with parameters from Lemma VII.16 – it is the topology generated by the two species of basic opens

$$[\![\, x = m \,]\!]_{X_0} = \left\{ \langle n, M\rangle \;\middle|\; \begin{array}{l} \mathfrak{K}_c \twoheadrightarrow M(c) \\ \text{sends } m \text{ to } n \end{array} \right\} \subseteq \coprod_{M\in X_0} M(c)$$

and

$$[\![\, U \,]\!]_{X_0} = \coprod_{M\in X_0} [\![\, U \,]\!]_M \subseteq \coprod_{M\in X_0} M(c).$$

Condition (ii) is automatically satisfied, and it is easily shown that π_c is a local homeomorphism for this topology. Finally, the functoriality condition (iii) follows from the naturality of $[\![\, - \,]\!]_{X_0}$ and the indexing (VII.iii). Namely we have that, for any arrow $d \xrightarrow{f} c$ of $\mathcal{C}$, ${f_{X_0}}^{-1}([\![\, U \,]\!]_{X_0}) = [\![\, P(f)(U) \,]\!]_{X_0}$ and $f_{X_0}^{-1}([\![\, x = m \,]\!]_{X_0}) = [\![\, y = f^{\mathfrak{K}}(m) \,]\!]_{X_0}$. Therefore, any topology τ_0 on X_0 containing τ-$\log_0$ admits a factorisation

$$\mathbf{Sets}^{X_0} \longrightarrow \mathbf{Sh}(X_0^{\tau_0}) \longrightarrow \mathbf{Sh}\left(X_0^{\tau\text{-}\log_0}\right) \dashrightarrow \mathcal{E}_P.$$

Now supposing the converse – that there is a choice of topologies T_c on $\coprod_{M\in X_0} M(c)$, for each $c \in \mathcal{C}$, satisfying conditions (i) to (iii), we identify an indexing of the models in X_0 for which τ_0 contains the opens $[\![\, m : U \,]\!]_{X_0}$. Just as in Proposition VII.17, we choose to index a element $n \in M(c)$ by the local sections $s: U \to \coprod_{M\in X_0} M(c)$ of $\pi_c: \coprod_{M\in X_0} M(c) \to X_0$ with open domain and open image, i.e. the open local sections of π_c, for which n is in the image $s(U)$. Since π_c is a local homeomorphism, this defines a partial surjection

$$\{\, \text{open local sections of } \pi_c \,\} \twoheadrightarrow M(c).$$

It remains to show that this indexing is natural in sense expressed in (VII.iii).

This follows since, for each arrow $d \xrightarrow{f} c \in \mathcal{C}$, and each open local section s of π_d, the composite $f_{X_0} \circ s$ is an open local section of π_c. As the triangle

$$\begin{tikzcd}
\left(\coprod_{M\in X_0} M(d)\right)^{T_d} \arrow[rr, "f_{X_0}"] \arrow[dr, "\pi_c"'] & & \left(\coprod_{M\in X_0} M(c)\right)^{T_c} \arrow[dl, "\pi_d"] \\
& X_0^{\tau_0} &
\end{tikzcd}$$

commutes, f_{X_0} is an open map by [63, Lemma C1.3.2] and so $f_{X_0} \circ s$ is still an open map, and secondly $f_{X_0} \circ s$ is evidently a section of π_c since

$$\pi_c \circ f_{X_0} \circ s = \pi_d \circ s = \mathrm{id}_{X_0}.$$

Therefore, for this indexing, the topology T_c contains as opens $s(U) = [\![\, x = s \,]\!]_{X_0}$ and $[\![\, U \,]\!]_{X_0}$. Thus, since the local homeomorphism π_c is, in particular, an open map, τ_0 contains the open

$$\pi_c(s(U) \cap [\![\, U \,]\!]_{X_0}) = [\![\, m : U \,]\!]_{X_0}$$

and hence the logical topology. $\square$

The other results from Sections VII.3 to VII.4 generalise without difficulty. Thus, as in Proposition VII.24 and Corollary VII.32, we arrive at the fact that, for a doctrinal site (P, J) and a groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ of models of P, the groupoid can be endowed with topologies making it a representing open topological groupoid if and only if the canonical localic geometric morphism

$$\mathbf{p}^{\log} \colon \mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \longrightarrow \mathbf{Sh}(C \rtimes P, J) \simeq \mathbf{Sh}(\mathfrak{Z}(P, J)) \simeq \mathcal{E}_P, \tag{VII.iv}$$

yields an equivalence of topoi $\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right) \simeq \mathcal{E}_P$, where

(i) the topology $\tau\text{-}\log_0$ on X_0 is generated by basic opens $[\![\, m : U \,]\!]_{\mathbb{X}}$, for each $c \in C$, $m \in \mathfrak{R}_c$ and $U \in P(c)$, as in Proposition VII.71,

(ii) and the topology $\tau\text{-}\log_1$ on X_1 generated by basic opens

$$\left[\!\!\left[\begin{matrix} m_1 : U, \\ m_2 \mapsto m_3, \\ m_4 : V \end{matrix}\right]\!\!\right]_{\mathbb{X}} = \left\{ M \xrightarrow{\alpha} N \in X_1 \;\middle|\; \begin{matrix} m_1 \in [\![\, U \,]\!]_M, \\ \alpha_d(m_2) = m_3, \\ m_4 \in [\![\, V \,]\!]_N \end{matrix} \right\},$$

where $c, d, e \in C$, $m_1 \in \mathfrak{R}_c$, $U \in P(c)$, $m_2, m_3 \in \mathfrak{R}_d$, $m_4 \in \mathfrak{R}_e$ and $V \in P(e)$.

Hence, just as in Section VII.4, the problem of identifying a representing open topological groupoid resolves down to identifying when the morphism between internal locales

$$\mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right)}\left(\textstyle\coprod_{M \in X_0} M(-)\right) \longrightarrow \mathfrak{Z}(P, J),$$

corresponding to (VII.iv), whose component at $c \in C$ is the frame homomorphism

$$\begin{aligned} [\![\, - \,]\!]_{\mathbb{X}_c}^{-1} \colon \mathfrak{Z}(P, J) &\to \mathrm{Sub}_{\mathbf{Sh}\left(\mathbb{X}^{\tau\text{-}\log_1}_{\tau\text{-}\log_0}\right)}\left(\coprod_{M \in X_0} M(c)\right), \\ S &\mapsto [\![\, S \,]\!]_{\mathbb{X}}, \end{aligned}$$

is an isomorphism of internal locales in $\mathbf{Sets}^{C^{\mathrm{op}}}$. We can easily identify conditions corresponding to injectivity and surjectivity of the frame homomorphism $[\![\, - \,]\!]_{\mathbb{X}_c}^{-1}$ that generalise the conditions of conservativity and elimination of parameters identified for model groupoids of geometric theories in Theorem VII.8, thus yielding our characterisation of the representing open topological groupoids of a doctrinal site:

Theorem VII.72 (Classification of representing open topological groupoids for doctrinal sites)**.** *Let $P \colon C^{\mathrm{op}} \to$ **PreOrd** be a doctrine whose desired set-based models are encoded by a Grothendieck topology J on $C \rtimes P$, i.e.*

$$P\text{-mod}(\mathbf{Sets}) \simeq \mathbf{DocSites}((P, J), (\mathscr{P}, K_{\mathscr{P}})).$$

*Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a small subgroupoid of P-*mod(**Sets**)*. The following are equivalent.*

(i) There exist topologies on X_1 *and* X_0 *making* $\mathbb{X} = (X_1 \rightrightarrows X_0)$ *an open topological groupoid for which there is an equivalence of topoi*

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(C \rtimes P, J) \simeq \mathcal{E}_P.$$

(ii) The groupoid $\mathbb{X}$ *satisfies the following two conditions.*

a) The groupoid $\mathbb{X}$ *is* J-conservative, *by which we mean that, for all* $U, V \in P(c)$, *if*

$$[\![U]\!]_M = [\![V]\!]_M$$

for all $M \in X_0$, *then there is a set of pairs*

$$\left\{ (f_i, W_i) \,\middle|\, d_i \xrightarrow{f_i} c \in C,\ W_i \leqslant P(f_i)(U), P(f_i)(V) \right\}$$

such that both of the resultant families of arrows in $C \rtimes P$

$$\left\{ (d_i, W_i) \xrightarrow{f_i} (c, U) \,\middle|\, i \in I \right\} \text{ and } \left\{ (d_i, W_i) \xrightarrow{f_i} (c, V) \,\middle|\, i \in I \right\}$$

are J-covering, or equivalently $\eta_c^{(P,J)}(U) = \eta_c^{(P,J)}(V)$.

b) There exists an indexing of $\mathbb{X}$ *by parameters* $\mathfrak{K} \colon C^{\mathrm{op}} \to \mathbf{Sets}$ *for which* $\mathbb{X}$ geometrically eliminates parameters, *by which we mean that, for each of parameter* $m \in \mathfrak{K}_c$, *there is some* $S \in \mathfrak{Z}(P, J)(c)$ *such that*

$$\begin{aligned}\overline{[\![x = m]\!]_{\mathbb{X}}} &= \left\{ \langle n, N \rangle \,\middle|\, \exists M \xrightarrow{\alpha} N \in X_1 \text{ such that } \alpha(m) = n \right\}, \\ &= [\![S]\!]_{\mathbb{X}} = \bigcup_{(f,U) \in S} f_{X_0}\left(\coprod_{M \in X_0} [\![U]\!]_M \right).\end{aligned}$$

Example VII.73. Let $\mathbb{T}$ be a theory over a fragment of logic that interprets regular logic, e.g. $\mathbb{T}$ could be a coherent or a classical theory. Note that since the associated doctrine $F^{\mathbb{T}}$ is still fibred over the category $\mathbf{Con}_N$, by Remark VII.70, the notion of an indexed model of $F^{\mathbb{T}}$ given in Definition VII.69 coincides with the usual notion offered in Definition VII.1.

Considering how the geometric completion of an existential doctrinal site is computed (see Proposition IV.6), we recognise that an indexed groupoid of $\mathbb{T}$-models $\mathfrak{K} \twoheadrightarrow \mathbb{X}$ eliminates parameters if and only if, for each tuple of parameters $\vec{m} \in \mathfrak{K}$, there is a set $\{ \varphi_i \,|\, i \in I \}$ of formulae in context $\vec{x}$ (over the appropriate fragment of logic, e.g. classical formulae if $\mathbb{T}$ is a classical theory) for which

$$\overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}} = \bigcup_{i \in I} [\![\vec{x} : \varphi_i]\!]_{\mathbb{X}}.$$

Chapter VIII

Weak equivalences of groupoids

Geometric morphisms as generalised continuous maps. The results of [68], [17] express that topoi (with enough points) can be thought of as 'spaces whose points can have non-trivial isomorphisms', in either a pointset or pointfree sense. Given this perspective on the objects of **Topos**, it would complete the intuition if geometric morphisms, the arrows of **Topos**, could also be thought of as 'continuous maps of spaces that respect isomorphisms of points'.

There is already a sense expressed in the literature in which this is true in the localic setting. Because multiple localic groupoids may represent the same topos, it is unsurprising that we must employ *weak equivalences* and (bi)categories of fractions, as introduced in [40] and extended to the bicategorical setting in [102]. Informally, a category of fractions is obtained by quotienting the arrows of the category by a class of arrows that 'should' be isomorphisms. In [92, §7], Moerdijk demonstrated an equivalence between the category of topoi **Topos** and the category of étale complete localic groupoids and their *homomorphisms*, localised by a *right* calculus of fractions.

Key result. Our aim in this chapter is to establish a topological parallel in the biequivalence

$$\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}, \tag{VIII.i}$$

where

(i) $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ is the bicategory of topoi with enough points, geometric morphisms, and isomorphisms between these,

(ii) **LogGrpd** is a bicategory of topological groupoids,

(iii) and $\mathfrak{W}$ is a *left* bicalculus of fractions on **LogGrpd**.

This gives a sense in which every geometric morphism between topoi with enough points is a 'continuous map that respects isomorphisms of points', but in a pointset rather than pointfree setting. Because the biequivalence (VIII.i) involves a left bicalculus of fractions while the equivalence established in [92] uses a right calculus, our biequivalence has a notably different flavour.

Logical motivation. Why is the biequivalence (VIII.i) of interest to the logician? Recall from Section VI.1 that, given a pair of geometric theories $\mathbb{T}$ and $\mathbb{T}'$, any geometric morphism between classifying topoi

$$f\colon \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}'}$$

yields a functor

$$F\colon \mathbb{T}\text{-mod}(\mathcal{F}) \longrightarrow \mathbb{T}'\text{-mod}(\mathcal{F})$$

that is natural in $\mathcal{F}$, and vice versa. We can think of the latter as instructions on how to transform a $\mathbb{T}$-model into a $\mathbb{T}'$-model. The biequivalence (VIII.i) expresses that, up to a weak equivalence, such transformations can be detected on the level of representing groupoids, lending credence to the intuition that a representing model groupoid is one that 'possesses enough information' recover the whole theory.

Overview. The chapter proceeds as follows.

(A) We begin in Section VIII.1 by recalling Moerdijk's equivalence from [92, §7]. In contrast to the localic setting, because there exist geometric morphisms that are not surjective on points, we cannot obtain a similar biequivalence for topoi with enough points using right calculi on topological groupoids, as demonstrated in Proposition VIII.4.

(B) Having negated the possibility of obtaining a biequivalence with a right calculus of fractions, the biequivalence (VIII.i) is established in Section VIII.2.

VIII.1 Topoi as a right category of fractions

Bifunctoriality of the topos of sheaves construction. Recall from [92, §5] that sending a localic groupoid to its topos of sheaves is a bifunctorial construction with respect to homomorphisms of localic groupoids and their transformations.

Definition VIII.1 (Definition 4.1 [92])**.** A *homomorphism of localic groupoids* $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ is a pair of locale morphisms $\varphi_0\colon X_0 \to Y_0$ and $\varphi_1\colon X_1 \to Y_1$, between the locales of objects and arrows respectively, which commute with the respective structure morphisms of the groupoids as in the diagram

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, "\varphi_1 \times_{\varphi_0} \varphi_1"] \arrow[d, "m"'] & Y_1 \times_{Y_0} Y_1 \arrow[d, "m'"] \\
X_1 \arrow[r, "\varphi_1"] \arrow[d, shift right=2, "s"'] \arrow[d, shift left=2, "t"] & Y_1 \arrow[d, shift right=2, "s'"'] \arrow[d, shift left=2, "t'"] \\
X_0 \arrow[u, "e" description] \arrow[r, "\varphi_0"'] & Y_0 . \arrow[u, "e'" description]
\end{tikzcd} \tag{VIII.ii}$$

This is precisely what it means to be a functor between internal categories.

Each homomorphism of localic groupoids $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ induces a geometric morphism $\mathbf{Sh}(\varphi)\colon \mathbf{Sh}(\mathbb{X}) \to \mathbf{Sh}(\mathbb{Y})$. The inverse image functor $\mathbf{Sh}(\varphi)^*$ sends descent datum $(W, \theta) \in \mathbf{Sh}(\mathbb{Y})$ to the pair consisting of $\varphi_0^*(W)$ and the map

$$s^*\varphi_0^*(W) = \varphi_1^* s'^*(W) \xrightarrow{\varphi_1^*(\theta)} \varphi_1^* t'^*(W) = t^*\varphi_0^*(W).$$

That $\varphi_1^*(\theta)$ satisfies the required equations to define descent datum on $\varphi_0^*(W)$ follows from the commutativity of (VIII.ii). Each morphism $(W,\theta) \xrightarrow{f} (W',\theta')$ of descent data is sent by $\mathbf{Sh}(\varphi)^*$ to the map $(\varphi_0^*(W), \varphi_1^*(\theta)) \xrightarrow{\varphi_0^*(f)} (\varphi_0^*(W'), \varphi_1^*(\theta'))$. The required commutativity condition $\varphi_1^*(\theta') \circ s^*\varphi_0^*(f) = t^*\varphi_0^*(f) \circ \varphi_1^*(\theta)$ follows, since

$$\begin{aligned}\varphi_1^*(\theta') \circ s^*\varphi_0^*(f) &= \varphi_1^*(\theta') \circ \varphi_1^* s'^*(f) \\ &= \varphi_1^*(\theta' \circ s'^*(f)) \\ &= \varphi_1^*(t'^*(f) \circ \theta) \\ &= \varphi_1^* t'^*(f) \circ \varphi_1^*(\theta) \\ &= t^*\varphi_0^*(f) \circ \varphi_1^*(\theta).\end{aligned}$$

Remark VIII.2. In this chapter, we are following [92] in that we are using homomorphisms of localic/topological groupoids to induce geometric morphisms. In a later paper [93, §4-5], Moerdijk shows that it is also possible to use *bispaces*.

Definition VIII.3. Given a pair of parallel homomorphisms

$$\mathbb{X} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathbb{Y}$$

between localic groupoids, a *transformation* $\varphi \overset{a}{\Rightarrow} \psi$ is a locale morphism $a\colon X_0 \to Y_1$ such that

(i) the equations $s' \circ a = \varphi_0$ and $t' \circ a = \psi_0$ are satisfied, expressing that a sends a (generalised) point $x \in X_0$ to its component $\varphi_0(x) \xrightarrow{a(x)} \psi(x) \in Y_1$,

(ii) and the square

$$\begin{array}{ccc} X_1 & \xrightarrow{(\psi_1, a\circ s)} & Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle (a\circ t, \varphi_1)}\downarrow & & \downarrow{\scriptstyle m'} \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow{m'} & Y_1 \end{array}$$

commutes, expressing that the choice of components is natural.

This is precisely what it means to be an internal natural transformation.

While it is clear how to define identity homomorphisms and composite homomorphisms for localic groupoids, it is perhaps less evident how to define a categorical structure on the transformations.

(i) The identity transformation $\varphi \overset{\mathrm{id}_\varphi}{\Longrightarrow} \varphi$ is named by the composite locale morphism

$$X_0 \xrightarrow{\varphi_0} Y_0 \xrightarrow{e'} Y_1,$$

(ii) while the composite of two transformations $\varphi \overset{a}{\Rightarrow} \psi$, $\psi \overset{a'}{\Rightarrow} \chi$ is given by the locale morphism

$$X_0 \xrightarrow{a \times_{Y_0} a'} Y_1 \times_{Y_0} Y_1 \xrightarrow{m'} Y_1.$$

Together, localic groupoids, their homomorphisms and transformations of these, yield a bicategory which we denote by **LocGrpd**.

A transformation $\varphi \overset{a}{\Rightarrow} \psi$ induces a natural transformation between the inverse image functors $\mathbf{Sh}(a)\colon \mathbf{Sh}(\varphi)^* \Rightarrow \mathbf{Sh}(\psi)^*$. The component of $\mathbf{Sh}(a)$ at the descent datum $(W, \theta) \in \mathbf{Sh}(\mathbb{Y})$ is given by the morphism

$$\varphi^*(Y, \theta) \xrightarrow{\mathbf{Sh}(a)_{(Y,\theta)}} \psi^*(Y, \theta)$$

as induced by the universal property of $\psi_0^*(Y)$ in the diagram

$$\begin{tikzcd}
\varphi_0^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrrr, dashed, bend left, "\mathbf{Sh}(a)_{(Y,\theta)}"] & s^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\theta"] & Y \arrow[d] & t^*(Y) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \psi_0^*(Y) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
G_0 \arrow[r, "a"] \arrow[rr, bend right, "\varphi_0"'] & H_1 \arrow[r, "s"] \arrow[rr, bend right, "i"'] & H_0 & H_1 \arrow[l, "t"'] & G_0 \arrow[l, "a"'] \arrow[ll, bend left, "\psi_0"]
\end{tikzcd}$$

since $s \circ a = \varphi_0$ and $t \circ a = \psi_0$ and every square is a pullback. In summary, there is a bifunctor

$$\mathbf{Sh}\colon \mathbf{LocGrpd} \longrightarrow \mathbf{Topos}.$$

All natural transformations between standard (set-based) groupoids are invertible. The same is true for transformations between homomorphisms. Let

$$\mathbb{X} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathbb{Y}$$

be a pair of parallel homomorphisms of localic groupoids, and let $\varphi \overset{a}{\Rightarrow} \psi$ be a transformation. It is easily shown, using a pointed argument (see Remark V.18), that the inverse to a is given by the composite locale morphism

$$X_0 \xrightarrow{a} Y_1 \xrightarrow{i'} Y_1.$$

Therefore, the bifunctor $\mathbf{Sh}\colon \mathbf{LocGrpd} \to \mathbf{Topos}$ factors through the bisubcategory $\mathbf{Topos}^{\mathrm{iso}}$, the bicategory of topoi, geometric morphisms and invertible 2-cells between these.

Moerdijk's equivalence. Recall from [68] or Theorem VI.10 that every topos is represented by an étale complete open localic groupoid. Thus, the restriction

$$\mathbf{ECG} \hookrightarrow \mathbf{LocGrpd} \xrightarrow{\mathbf{Sh}} \mathbf{Topos}^{\mathrm{iso}}$$

to the 1-full subcategory $\mathbf{ECG} \subseteq \mathbf{LocGrpd}$ of étale complete open localic groupoids is essentially surjective. It can moreover be shown that it is faithful on 1-cells. However, it is not full on 1-cells, and so the functor does not witness a biequivalence of bicategories.

Instead, as shown in [92, §7] and the bicategorical extension in [102], a biequivalence

$$\mathbf{ECG}[W^{-1}] \simeq \mathbf{Topos}$$

can be obtained by quotienting **ECG** by a *right bicalculus of fractions* W on the category **ECG**. In particular, for every geometric morphism

$$f \colon \mathbf{Sh}(\mathbb{X}) \longrightarrow \mathbf{Sh}(\mathbb{Y}),$$

there is a span of étale complete open localic groupoids

$$\begin{tikzcd} \mathbb{W} \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \mathbb{Y} \\ \mathbb{X} & \end{tikzcd}$$

such that $\mathbb{W} \xrightarrow{\psi} \mathbb{X} \in W$ and the triangle

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{W}) \arrow[r, "\mathbf{Sh}(\varphi)"] \arrow[d, "\mathbf{Sh}(\psi)"', "\wr"] & \mathbf{Sh}(\mathbb{Y}) \\ \mathbf{Sh}(\mathbb{X}) \arrow[ur, "f"'] & \end{tikzcd}$$

commutes and $\mathbf{Sh}(\psi)$ is an equivalence of topoi.

The homomorphisms $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y} \in W$, the *weak equivalences*, are those sent by **Sh** to equivalences of topoi, and these can be characterised as those homomorphisms that are open and, in the pointfree sense, essentially surjective and fully faithful. That is,

(i) the locale morphisms φ_0 and φ_1 are open,

(ii) the composite

$$X_0 \times_{Y_0} Y_1 \xrightarrow{\mathrm{pr}_2} Y_1 \xrightarrow{t'} Y_0$$

is an open surjection, expressing that φ is essentially surjective,

(iii) the square

$$\begin{tikzcd} X_1 \arrow[r, "\varphi_1"] \arrow[d, "(s,t)"'] & Y_1 \arrow[d, "(s',t')"] \\ X_0 \times X_0 \arrow[r, "\varphi_0 \times \varphi_0"] & Y_0 \times Y_0 \end{tikzcd}$$

is a pullback of locales, expressing that φ is fully faithful.

VIII.1.1 Right calculi of fractions on topological groupoids

Just as with localic groupoids, taking the topos of sheaves on a topological groupoid is bifunctorial with respect to homomorphisms and transformations of topological groupoids. These are defined by replacing 'locale' with 'topological space' in Definition VIII.1 and Definition VIII.3. Alternatively,

(i) a homomorphism $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ of topological groupoids is a functor on the underlying groupoids such that the action on objects $\varphi_0\colon X_0 \to Y_0$ and the action on arrows $\varphi_1\colon X_1 \to Y_1$ are both continuous,

(ii) a transformation $\varphi \overset{a}{\Rightarrow} \psi$ between a pair of parallel homomorphisms of topological groupoids is a natural transformation of the underlying functors of φ and ψ such that the map $a\colon X_0 \to Y_1$ that sends an object to its component is continuous.

As before, there is a bifunctor

$$\mathbf{TopGrpd} \xrightarrow{\mathbf{Sh}} \mathbf{Topos}^{\mathrm{iso}}.$$

This bifunctor factors through the 1-full 2-subcategory $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \subseteq \mathbf{Topos}^{\mathrm{iso}}$, the 2-subcategory of topoi with enough points. The result of Butz and Moerdijk [17] (see also Section VII.5.4) expresses that the functor $\mathbf{Sh}$ is essentially surjective.

We may therefore wonder if, in analogy with localic groupoids, there is a biequivalence between $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ and a category of right fractions on a suitable 2-subcategory of $\mathbf{TopGrpd}$. This is, however, never possible.

Proposition VIII.4. *For any 2-subcategory $C \subseteq \mathbf{TopGrpd}$, and any bicalculus of right fractions Σ on C,*

$$\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \not\simeq C[\Sigma^{-1}].$$

Proof. We construct an example of a geometric morphism that cannot be obtained by a span of topological groupoids. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a topological groupoid contained in $C \subseteq \mathbf{TopGrpd}$ such that the topos $\mathbf{Sh}(\mathbb{X})$ has a point $p\colon \mathbf{Sets} \to \mathbf{Sh}(\mathbb{X})$ that does not correspond to a point of X_0, i.e. there is no factorisation

$$\begin{tikzcd} & \mathbf{Sh}(X_0) \arrow[d, "u_{\mathbb{X}}"] \\ \mathbf{Sets} \arrow[ur, dashed, "/" marking] \arrow[r, "p"'] & \mathbf{Sh}(\mathbb{X}), \end{tikzcd}$$

where $u_{\mathbb{X}}\colon \mathbf{Sh}(X_0) \to \mathbf{Sh}(\mathbb{X})$ is the 'forgetting action' geometric morphism from Section V.1.1. For example, $\mathbf{Sh}(\mathbb{X})$ could be the classifying topos for a theory with unboundedly many models, which ensures the existence of such a p for any X_0.

Suppose that there is a biequivalence $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \simeq C[\Sigma^{-1}]$. Then there is a homomorphism of topological groupoids $\mathbb{Y} \xrightarrow{\varphi} \mathbb{X} \in C$ such that

$$\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sets} \xrightarrow{\mathbf{Sh}(\varphi)\simeq p} \mathbf{Sh}(\mathbb{X}).$$

Consequently, there is a commutative square of geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(Y_0) \arrow[r, "\mathbf{Sh}(\varphi_0)"] \arrow[d, "u_{\mathbb{Y}}"'] & \mathbf{Sh}(X_0) \arrow[d, "u_{\mathbb{X}}"] \\ \mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sets} \arrow[r, "p"] & \mathbf{Sh}(\mathbb{X}). \end{tikzcd}$$

Any point of the space Y_0 yields a section of $u_{\mathbb{Y}}$, as in the diagram

$$\begin{tikzcd} \mathbf{Sh}(Y_0) \arrow[r, "\mathbf{Sh}(\varphi_0)"] \arrow[d, "u_{\mathbb{Y}}"'] & \mathbf{Sh}(X_0) \arrow[d, "u_{\mathbb{X}}"] \\ \mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sets} \arrow[r, "p"] \arrow[u, dashed, bend right] & \mathbf{Sh}(\mathbb{X}). \end{tikzcd}$$

But such a point would yield a factorisation of p through $u_{\mathbb{X}}$, a contradiction. So we conclude that Y_0 is the empty space. But then, by Lemma V.8, there would be a surjective geometric morphism

$$\mathbf{0}_{\mathbf{Topos}} \simeq \mathbf{Sh}(Y_0) \xrightarrow{u_{\mathbb{Y}}} \mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sets},$$

but no such surjection exists[1]. Hence, $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \not\simeq C[\Sigma^{-1}]$. □

Nonetheless, the bifunctor $\mathbf{Sh}\colon \mathbf{TopGrpd} \to \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ does induce a biequivalence if we also restrict to suitable 2-subcategories of $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$. For example, Pronk establishes in [102, Theorem 27] a biequivalence

$$\mathbf{\acute{E}tendue}^{\mathrm{iso}}_{\mathrm{sp}} \simeq \mathbf{\acute{E}taleGrpd}[W^{-1}]$$

between the bicategory of (spatial) *étendues* (see [3, §IV.9.8.2(e)]) and a localisation of the bicategory of *étale* topological groupoids, i.e. those groupoids whose source and target maps are étale/local homeomorphisms.

VIII.2 Topoi as a left category of fractions

Although, as expressed in Proposition VIII.4, we cannot hope to represent the entire bicategory $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ by a bicalculus of right fractions on a 2-subcategory of **TopGrpd**, we can establish an equivalence if we instead consider a left bicategory of fractions.

In this section, we identify a 1-full 2-subcategory of **TopGrpd**, which we tentatively denote by **LogGrpd** and call the objects *logical groupoids*, and a bicalculus of *left* fractions $\mathfrak{W}$ on **LogGrpd** for which there is a biequivalence

$$\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}.$$

In particular, the biequivalence $\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}$ would entail that, given two logical groupoids $\mathbb{X}, \mathbb{Y} \in \mathbf{LogGrpd}$, there is an equivalence $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y})$ if and only if there is a cospan of homomorphisms

$$\begin{tikzcd} \mathbb{X} \arrow[dr, "\varphi"'] & & \mathbb{Y} \arrow[dl, "\psi"] \\ & \mathbb{W} & \end{tikzcd}$$

where $\mathbb{X} \xrightarrow{\varphi} \mathbb{W}, \mathbb{Y} \xrightarrow{\psi} \mathbb{W} \in \mathfrak{W}$. Recall from Chapter VII that, given some theory $\mathbb{T}$ classified by the topos $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{W})$, the groupoids $\mathbb{X}, \mathbb{Y}, \mathbb{W}$ can be associated

[1] Of course, here we are assuming that $\mathbf{Sets} \not\simeq \mathbf{0}_{\mathbf{Topos}}$, i.e. our chosen model of set theory is consistent.

with groupoid of models of $\mathbb{T}$ which think of as 'containing enough information to recover the theory'. With this perspective, it is therefore not too surprising that given two groupoids of models $\mathbb{X}, \mathbb{Y}$ 'with enough information', then we can expand them to a third model groupoid $\mathbb{W}$ which also has 'enough information'.

VIII.2.1 Logical groupoids

We first identify the 2-subcategory **LogGrpd** $\subseteq$ **TopGrpd**. Recall (from [79, §III.9], say) that, given a group G and two topologies τ_1 and τ_2 on G for which G^{τ_1}, G^{τ_2} are both topological groups, G^{τ_1} and G^{τ_2} are *Morita equivalent*, i.e. $\mathbf{B}G^{\tau_1} \simeq \mathbf{B}G^{\tau_2}$, if and only if τ_1 and τ_2 share the same open subgroups. For example, the group of the rationals $\mathbb{Q}$ with addition is a topological group for either the Euclidean topology or the co-discrete topology, but both have a unique open subgroup, and so there is an equivalence of topoi $\mathbf{B}\mathbb{Q} \simeq \mathbf{B}\mathbb{Q}^{\mathcal{E}} \simeq \mathbf{Sets}$. Logical groupoids intend to capture the same behaviour. They can also be compared to the *powder monoids* of [104, Definition 5.2.16 & Theorem 5.2.18].

Definition VIII.5. A topological groupoid $\mathbb{X}^{\tau_1}_{\tau_0} = (X_1^{\tau_1} \rightrightarrows X_0^{\tau_0})$ is said to be a *logical groupoid* if it is open, the spaces $X_0^{\tau_0}$ and $X_1^{\tau_1}$ are sober, and $\mathbb{X}^{\tau_1}_{\tau_0}$ is *étale complete* in the sense that:

(i) for a pair $x, y \in X_0$, any isomorphism of the corresponding points

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, "x"] \arrow[d, "y"'] & \mathbf{Sh}(X_0) \arrow[d, "u"] \arrow[dl, Rightarrow, "\alpha", "\sim"'] \\ \mathbf{Sh}(X_0) \arrow[r, "u"'] & \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}) \end{tikzcd}$$

is instantiated by an arrow $x \xrightarrow{\alpha} y \in X_1$,

(ii) and for any other topology τ_1' on X_1, if $\mathbb{X}^{\tau_1'}_{\tau_0}$ is a topological groupoid with

$$\mathbf{Sh}\left(\mathbb{X}^{\tau_1'}_{\tau_0}\right) \simeq \mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0}),$$

then $\tau_1' \supseteq \tau_1$. That is, τ_1 is the coarsest topology on X_1 determined by the topos $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$.

We denote by **LogGrpd** the 1-full 2-subcategory of **TopGrpd** of logical groupoids.

Remark VIII.6. Why have we suggested the name logical groupoid? Let $\mathbb{X}^{\tau_1}_{\tau_0}$ be a logical groupoid according to Definition VIII.5. Let $\mathbb{T}$ be a geometric theory classified by the topos $\mathbf{Sh}(\mathbb{X}^{\tau_1}_{\tau_0})$. In fact, we can require that $\mathbb{T}$ is an inhabited theory. To see why, note that any theory $\mathbb{T}'$ is Morita equivalent to the same theory with an added inhabited 'dummy' sort, i.e. the expansion of $\mathbb{T}'$ by a sort A, a constant c of type A, and the axioms $\top \vdash_{x:A} c = x$. We can then apply [63, Lemma D1.4.13] to deduce that $\mathbb{T}'$ is Morita equivalent to a theory with a (single) necessarily inhabited sort.

By Theorem VII.8, the groupoid $\mathbb{X}$ can be identified with an indexed groupoid of $\mathbb{T}$-models that is conservative and eliminates parameters. By Remark VII.18, Proposition VII.22 and the fact that $\mathbb{X}^{\tau_1}_{\tau_0}$ satisfies Definition VIII.5, we deduce that

τ_0 and τ_1 are the logical topologies for this indexed groupoid of $\mathbb{T}$-models. Moreover, since $\mathbb{X}$ satisfies the étale completeness condition Definition VIII.5(i), then $\mathbb{X}$ is also étale complete in the sense of Definitions VII.48.

Conversely, given a groupoid $\mathbb{X}$ of indexed Σ-structures that eliminates parameters and which is étale complete according to Definitions VII.48, by Proposition VII.22 and Lemma VII.28, the corresponding topological groupoid $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ is a logical groupoid according to Definition VIII.5. Hence, a topological groupoid is a logical groupoid if and only if it is obtained from an étale complete, indexed groupoid of Σ-structures that eliminates parameters, for some signature Σ.

Lemma VIII.7. *For a pair $\mathbb{X}, \mathbb{Y}$ of logical groupoids, the functor on hom-categories*

$$\mathbf{Sh}\colon \mathbf{LogGrpd}(\mathbb{X}, \mathbb{Y}) \longrightarrow \mathbf{Topos}^{\mathrm{iso}}(\mathbf{Sh}(\mathbb{X}), \mathbf{Sh}(\mathbb{Y}))$$

is faithful.

Proof. Let $\varphi \overset{a}{\Rightarrow} \psi$ be a transformation. By sobriety, the component $\varphi_0(x) \xrightarrow{a(x)} \psi_0(x) \in Y_1$ at x corresponds to the composite 2-cell

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, "x"] & \mathbf{Sh}(X_0) \arrow[r, "u_{\mathbb{X}}"] & \mathbf{Sh}(\mathbb{X}) \arrow[r, bend left=50, "\mathbf{Sh}(\varphi)", ""{name=U, below}] \arrow[r, bend right=50, "\mathbf{Sh}(\psi)"', ""{name=D, above}] & \mathbf{Sh}(\mathbb{Y}). \arrow[from=U, to=D, Rightarrow, "\mathbf{Sh}(a)"] \end{tikzcd}$$

Hence, if $\mathbf{Sh}(a) = \mathbf{Sh}(a')$ for another transformation $\varphi \overset{a'}{\Rightarrow} \psi$, then $a(x) = a'(x)$ for all x, i.e. $a = a'$. $\square$

Proposition VIII.8. *Let $f\colon \mathcal{F} \to \mathcal{E}$ be a geometric morphism between topoi with enough points. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a representing logical groupoid for $\mathcal{F}$, i.e. there is an equivalence $\mathcal{F} \simeq \mathbf{Sh}(\mathbb{X})$. Then there is a representing logical groupoid $\mathbb{Y}$ of $\mathcal{E}$ and a homomorphism of topological groupoids $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ such that there is an isomorphism of geometric morphisms*

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{F} \xrightarrow{\mathbf{Sh}(\varphi) \simeq f} \mathcal{E} \simeq \mathbf{Sh}(\mathbb{Y}).$$

Proof. Let $\mathbb{T}$ be a geometric theory, over a signature Σ, classified by the topos $\mathcal{E}$. By [22, Theorem 7.1.5], the geometric morphism f is, up to isomorphism, induced by a *geometric expansion* $\mathbb{T}'$ of $\mathbb{T}$. That is, $\mathbb{T}'$ is a geometric theory over an expanded signature $\Sigma' \supseteq \Sigma$ (that potentially adds new sorts) which contains the axioms of $\mathbb{T}$, and there is an isomorphism of geometric morphisms

$$\mathcal{E}_{\mathbb{T}'} \simeq \mathcal{F} \xrightarrow{e^{\mathbb{T}'}_{\mathbb{T}} \simeq f} \mathcal{E} \simeq \mathcal{E}_{\mathbb{T}}.$$

By Theorem VII.8, the groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ is a conservative groupoid of $\mathbb{T}'$-models with an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$ such that $\mathbb{X}$ eliminates parameters, and moreover, X_0 and X_1 are endowed with the respective logical topologies induced by this indexing.

Each point $M \in X_0$, which corresponds to a model of $\mathbb{T}'$, therefore yields a model of $\mathbb{T}$ via the composite

$$\textbf{Sets} \xrightarrow{M} \mathcal{E}_{\mathbb{T}'} \xrightarrow{e_{\mathbb{T}}^{\mathbb{T}'}} \mathcal{E}_{\mathbb{T}}.$$

We denote this $\mathbb{T}$-model, the $\mathbb{T}$-reduct of M, by $M|_{\mathbb{T}}$. The underlying sets interpreting the sorts of $M|_{\mathbb{T}}$ are simply the sets interpreting those sorts contained in the unexpanded signature $\Sigma \subseteq \Sigma'$. Thus, each $\mathbb{T}$-reduct $M|_{\mathbb{T}}$ still has a $\mathfrak{K}$-indexing, and moreover each isomorphism $M \xrightarrow{\alpha} N$ of $\mathbb{T}'$-models automatically yields an isomorphism of the $\mathbb{T}$-reducts

$$M|_{\mathbb{T}} \xrightarrow{\alpha|_{\mathbb{T}}} N|_{\mathbb{T}}.$$

By Corollary VII.57(ii), there exists a choice of representing groupoid $\mathbb{Y} = (Y_1 \rightrightarrows Y_0)$ of $\mathcal{E} \simeq \mathcal{E}_{\mathbb{T}}$ where Y_0 contains an isomorphic copy of $M|_{\mathbb{T}}$ for each $\mathbb{T}'$-model $M \in X_0$. Note that the groupoid $\mathbb{Y}$ can be chosen as the Forssell groupoid of all $\mathfrak{K}'$-indexed $\mathbb{T}$-models $\mathcal{FG}(\mathfrak{K}')$ for some set of parameters where $\mathfrak{K}' \supseteq \mathfrak{K}$.

Therefore, by making some choice of $M|_{\mathbb{T}} \cong M' \in Y_0$ for each $M \in X_0$, we obtain a functor of the underlying groupoids $\varphi \colon \mathbb{X} \to \mathbb{Y}$,

$$M \in X_0 \mapsto M|_{\mathbb{T}} \cong M' \in Y_0,$$
$$\left[M \xrightarrow{\alpha} N\right] \in X_1 \mapsto \left[M' \cong M|_{\mathbb{T}} \xrightarrow{\alpha|_{\mathbb{T}}} N|_{\mathbb{T}} \cong N'\right] \in Y_1.$$

We must choose the models $M|_{\mathbb{T}} \cong M' \in Y_0$ in such a way that the map on objects $\varphi_0 \colon X_0 \to Y_0$ and the map on arrows $\varphi_1 \colon X_1 \to Y_1$ are continuous with respect to the respective logical topologies.

We obtain this by setting $\varphi_0(M)$ as the $\mathbb{T}$-reduct $M|_{\mathbb{T}}$ with the already present indexing $\mathfrak{K}' \supseteq \mathfrak{K} \twoheadrightarrow M|_{\mathbb{T}}$. It is easily checked that, thus defined, both φ_0 and φ_1 are continuous with respect to the logical topologies. That $\mathbf{Sh}(\varphi) \simeq f$ follows from Lemma VIII.7 since they agree (up to isomorphism) on the points of $\mathcal{F}$ and the isomorphisms of these points corresponding to the groupoid $\mathbb{X}$. □

Lemma VIII.9. *For a pair $\mathbb{X}, \mathbb{Y}$ of logical groupoids, the functor on hom-categories*

$$\mathbf{Sh} \colon \mathbf{LogGrpd}(\mathbb{X}, \mathbb{Y}) \longrightarrow \mathbf{Topos}^{\mathrm{iso}}(\mathbf{Sh}(\mathbb{X}), \mathbf{Sh}(\mathbb{Y}))$$

is also full.

Proof. Let $\varphi, \psi \colon \mathbb{X} \rightrightarrows \mathbb{Y}$ be a pair of homomorphisms, and let $\mathbf{Sh}(\varphi) \overset{\gamma}{\Rightarrow} \mathbf{Sh}(\psi)$ be an isomorphism between the induced geometric morphisms. Thus, by Definition VIII.5(i), $x \in X_0$, there is corresponding arrow $\varphi_0(x) \xrightarrow{\gamma_x} \psi_0(x) \in Y_1$. This defines a natural transformation $\gamma' \colon X_0 \to Y_1$ between the underlying functors of φ and ψ. It remains to show that γ' is continuous.

Just as in Proposition VIII.8, we can assume that $\mathbf{Sh}(\mathbb{Y})$ classifies a geometric theory $\mathbb{T}$, and φ and ψ are induced by geometric expansions $\mathbb{T}'$ and $\mathbb{T}''$ of $\mathbb{T}$, i.e. there is a diagram

$$\begin{tikzcd} \mathcal{E}_{\mathbb{T}'} \simeq \mathcal{E}_{\mathbb{T}''} \simeq \mathbf{Sh}(\mathbb{X}) \arrow[r, bend left, "\mathbf{Sh}(\varphi)\simeq e_{\mathbb{T}}^{\mathbb{T}'}"] \arrow[r, bend right, "\mathbf{Sh}(\psi)\simeq e_{\mathbb{T}}^{\mathbb{T}''}"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathbf{Sh}(\mathbb{Y}) \simeq \mathcal{E}_{\mathbb{T}}. \end{tikzcd}$$

Hence, $\mathbb{Y}$ can be associated with a conservative groupoid of $\mathfrak{K}$-indexed $\mathbb{T}$-models that eliminate parameters. Similarly $\mathbb{X}$ can simultaneously be identified with a representing groupoid of $\mathfrak{K}'$-indexed $\mathbb{T}'$-models and a representing groupoid of $\mathfrak{K}''$-indexed $\mathbb{T}''$-models, where we can also assume that $\mathfrak{K}', \mathfrak{K}'' \supseteq \mathfrak{K}$. The homomorphism φ (respectively, ψ) sends the $\mathbb{T}'$-model M (resp., $\mathbb{T}''$-model N) corresponding to an object $x \in X_0$ to its $\mathbb{T}$-reduct $M|_{\mathbb{T}}$ (resp., $N|_{\mathbb{T}}$) with the same $\mathfrak{K} \subseteq \mathfrak{K}'$-indexing (resp., $\mathfrak{K}''$-indexing). The transformation $\gamma' \colon X_0 \to Y_1$ sends each point $x \in X_0$ to an isomorphism of the corresponding reducts $M|_{\mathbb{T}} \cong N|_{\mathbb{T}}$.

By Remark VIII.6, we can assume that Y_1 is endowed with the logical topology for arrows and that X_0 is endowed with the logical topology for objects (for both the $\mathbb{T}'$ and $\mathbb{T}''$ logical structure with which it is associated). Thus, a point $x \in X_0$ is contained in the inverse image

$$\gamma'^{-1}\left(\left[\!\!\left[\begin{matrix} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{matrix}\right]\!\!\right]_{\mathbb{Y}}\right)$$

if and only if $M \vDash \varphi(\vec{a})$, $N \vDash \psi(\vec{d})$, and $\vec{b}, \vec{c}$ are identified in the $\mathbb{T}$-reducts $M|_{\mathbb{T}} \cong N|_{\mathbb{T}}$ (recall that $\mathfrak{K} \subseteq \mathfrak{K}', \mathfrak{K}''$ and the signatures of $\mathbb{T}', \mathbb{T}''$ expand that of $\mathbb{T}$, so all these conditions are well-defined). Therefore,

$$[\![\, \vec{a} : \varphi \,]\!]_{\mathbb{X}} \cap [\![\, \vec{d} : \psi \,]\!]_{\mathbb{X}} \cap [\![\, \vec{b} = \vec{c} \,]\!]_{\mathbb{X}} = \gamma'^{-1}\left(\left[\!\!\left[\begin{matrix} \vec{a} : \varphi \\ \vec{b} \mapsto \vec{c} \\ \vec{d} : \psi \end{matrix}\right]\!\!\right]_{\mathbb{Y}}\right),$$

and so γ' is indeed continuous. □

VIII.2.2 Weak equivalences of logical groupoids

We now turn to describing the weak equivalences of logical groupoids. As currently formulated, Definition VIII.10 relies on the logical approach to representing topological groupoids developed in Chapter VII. Following this, we are able to establish our desired biequivalence.

Definition VIII.10. A homomorphism of logical groupoids $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ is said to be a *weak equivalence* if the following are satisfied.

(i) There is a common choice of geometric theory $\mathbb{T}$ and a set of parameters $\mathfrak{K}$ such that $\mathbb{X}$ and $\mathbb{Y}$ are groupoids of $\mathfrak{K}$-indexed $\mathbb{T}$-models, endowed with the induced logical topologies, and moreover $\mathbb{X}$ is contained in $\mathbb{Y}$, i.e. there are inclusions of topological subgroupoids
$$\mathbb{X} \subseteq \mathbb{Y} \subseteq \mathcal{FG}(\mathfrak{K}).$$

(ii) Viewed as groupoids of indexed $\mathbb{T}$-models, both $\mathbb{X}$ and $\mathbb{Y}$ are conservative and eliminate parameters.

We denote the class of weak equivalences by $\mathfrak{W}$.

As an immediate corollary of Theorem VII.8, we obtain that:

Corollary VIII.11. *If* $\mathbb{Y} \xrightarrow{\psi} \mathbb{W}$ *is a weak equivalence of logical groupoids, then* $\mathbf{Sh}(\psi)$ *induces an equivalence* $\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{W})$*.*

We are almost ready to demonstrate the biequivalence

$$\mathbf{Topos}^{\mathrm{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}.$$

We must first show that $\mathfrak{W}$ defines a bicalculus of left fractions on **LogGrpd** according to [102].

Proposition VIII.12. *The class* $\mathfrak{W}$ *of weak equivalences defines a bicalculus of left fractions on* **LogGrpd***.*

Proof. Recall from [102, §2.1] that there are three conditions required to be a bicalculus of left fractions. The class $\mathfrak{W}$ must be wide, satisfy the left Ore condition, and the left cancellability condition (all in the bicategorical sense). We demonstrate each in turn.

(i) (Wideness) Evidently, $\mathfrak{W}$ contains all identities, is closed under composition, and if $\varphi \overset{a}{\Rightarrow} \psi$ is a (necessarily invertible) transformation with $\varphi \in \mathfrak{W}$, then ψ is contained in $\mathfrak{W}$ too. Thus, $\mathfrak{W}$ is wide.

(ii) (Left Ore condition) Let

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "\psi"] \arrow[d, "\varphi"'] & \mathbb{W} \\ \mathbb{X} & \end{tikzcd}$$

be a span where $\mathbb{Y} \xrightarrow{\psi} \mathbb{W}$ is a weak equivalence. We wish to find a square of homomorphisms

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\cong"] & \mathbb{W} \arrow[d, "\varphi'"] \\ \mathbb{X} \arrow[r, "\psi'"] & \mathbb{V} \end{tikzcd} \tag{VIII.iii}$$

that commutes up to isomorphism, where ψ' is a weak equivalence.

The homomorphism $\mathbb{Y} \xrightarrow{\varphi} \mathbb{X}$ induces a geometric morphism

$$\mathbf{Sh}(\mathbb{W}) \simeq \mathbf{Sh}(\mathbb{Y}) \xrightarrow{\mathbf{Sh}(\varphi)} \mathbf{Sh}(\mathbb{X}).$$

By applying Proposition VIII.8, there is a representing logical groupoid $\mathbb{V}$ of $\mathbf{Sh}(\mathbb{X})$ and a homomorphism of topological groupoids $\mathbb{W} \xrightarrow{\varphi'} \mathbb{V}$. By Corollary VII.57(ii), we can ensure also that $\mathbb{V}$ contains $\mathbb{X}$ as a subgroupoid in such a way that the square (VIII.iii) commutes up to isomorphism. The inclusion $\mathbb{X} \xrightarrow{\psi'} \mathbb{V}$ is, by construction, a weak equivalence.

(iii) (Left cancellability) Let

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\chi"] & \mathbb{Y} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \mathbb{W} \end{tikzcd}$$

be a fork of homomorphisms of logical groupoids that commute up to isomorphism, and where $\mathbb{X} \xrightarrow{\chi} \mathbb{Y} \in \mathfrak{W}$. We wish to find another homomorphism $\mathbb{W} \xrightarrow{\chi'} \mathbb{V} \in \mathfrak{W}$ such that the fork

$$\begin{tikzcd} \mathbb{Y} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \mathbb{W} \arrow[r, "\chi'"] & \mathbb{V} \end{tikzcd}$$

also commutes up to isomorphism.

Since the induced geometric morphisms

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y}) \arrow[r, shift left, "\mathbf{Sh}(\varphi)"] \arrow[r, shift right, "\mathbf{Sh}(\psi)"'] & \mathbf{Sh}(\mathbb{W}) \end{tikzcd}$$

also commute, i.e. $\mathbf{Sh}(\varphi) \simeq \mathbf{Sh}(\psi)$, by Lemma VIII.7, we have that $\varphi \simeq \psi$, and so we can simply take $\mathrm{id}_{\mathbb{W}}$ as the desired weak equivalence.

We must also show that, for any other weak equivalence of logical groupoids $\mathbb{W} \xrightarrow{\chi'} \mathbb{V} \in \mathfrak{W}$ that makes the fork

$$\begin{tikzcd} \mathbb{Y} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \mathbb{W} \arrow[r, "\chi'"] & \mathbb{V} \end{tikzcd}$$

commute up to an isomorphism, then there are a pair of weak equivalences $\mathbb{W} \xrightarrow{\rho} \mathbb{V}', \mathbb{V} \xrightarrow{\rho'} \mathbb{V}' \in \mathfrak{W}$ and a coherent choice of isomorphism $\rho \circ \mathrm{id}_{\mathbb{W}} \overset{\sim}{\Rightarrow} \rho' \circ \chi'$. But we can simply choose ρ as $\mathbb{W} \xrightarrow{\chi'} \mathbb{V} \in \mathfrak{W}$ and ρ' as $\mathbb{V} \xrightarrow{\mathrm{id}_{\mathbb{V}}} \mathbb{V} \in \mathfrak{W}$. It is trivial to show that the identity transformation $\chi' \circ \mathrm{id}_{\mathbb{W}} = \mathrm{id}_{\mathbb{V}} \circ \chi'$ satisfies the necessary coherence condition expressed in [102, §2.1].

Thus, $\mathfrak{W}$ defines a bicalculus of left fractions on **LogGrpd**. □

In [102], Pronk provides a bicategorical extension to Gabriel and Zisman's localisation result (see [40, Proposition I.1.3]).

Lemma VIII.13 (Proposition 24 [102]). *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a bifunctor and let Σ be a class of 1-morphisms in $\mathcal{C}$ admitting a left bicalculus of fractions. Suppose that*

(i) the bifunctor G is essentially surjective on objects and fully faithful on 2-cells,

(ii) for each $f \in W$, $G(f)$ is an equivalence,

(iii) and for any arrow $G(c) \xrightarrow{g} G(d) \in \mathcal{D}$, there are a pair of arrows $c \xrightarrow{f} e \in \mathcal{C}$ and $d \xrightarrow{\sigma} e \in \Sigma$ such that the triangle

$$\begin{tikzcd} G(c) \arrow[r, "g"] \arrow[dr, "G(f)"'] & G(d) \arrow[d, "G(\sigma)"] \\ & G(e) \end{tikzcd}$$

commutes up to isomorphism.

Then there is a biequivalence of bicategories $[\Sigma^{-1}]\mathcal{C} \simeq \mathcal{D}$.

Theorem VIII.14. *There is a biequivalence*

$$[\mathfrak{W}^{-1}]\mathbf{LogGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}.$$

Proof. The functor

$$\mathbf{Sh}\colon \mathbf{LogGrpd} \longrightarrow \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$$

is essentially surjective by [17] (see also Corollary VII.56) and fully faithful on 2-cells by Lemma VIII.7 and Lemma VIII.9. By Corollary VIII.11, **Sh** sends arrows in $\mathfrak{W}$ to equivalences of topoi.

Let $f\colon \mathbf{Sh}(\mathbb{X}) \to \mathbf{Sh}(\mathbb{Y})$ be a geometric morphism. By Proposition VIII.8, f is isomorphic to the geometric morphism $\mathbf{Sh}(\varphi)$ induced by a homomorphism of logical groupoids $\mathbb{X} \xrightarrow{\varphi} \mathbb{W}$, where $\mathbb{W}$ is a representing groupoid for the topos $\mathbf{Sh}(\mathbb{Y})$. By Corollary VII.57(ii), we can also choose $\mathbb{W}$ to contain $\mathbb{Y}$ as a subgroupoid, and moreover this inclusion $\mathbb{Y} \hookrightarrow \mathbb{W}$ can be chosen to be a weak equivalence. Hence, by Lemma VIII.13, there is the desired biequivalence $[\mathfrak{W}^{-1}]\mathbf{LocGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$. □

The logical interpretation. As a consequence of the biequivalence

$$[\mathfrak{W}^{-1}]\mathbf{LogGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.},$$

we are able to transform the problem of detecting Morita equivalences between logical theories into a problem of topological algebra, as promised earlier. The full biequivalence $[\mathfrak{W}^{-1}]\mathbf{LogGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$ is not necessary to deduce the following result; it could instead be proven as a consequence of Theorem VII.8 and Corollary VII.57 alone.

Corollary VIII.15. *Let $\mathbb{T}$ and $\mathbb{T}'$ be two geometric theories with representing groupoids of models $\mathbb{X}$ and $\mathbb{Y}$ respectively. The theories $\mathbb{T}$ and $\mathbb{T}'$ are Morita equivalent, meaning that there is an equivalence of topoi*

$$\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$$

if and only if there is a cospan of homomorphisms of logical groupoids

$$\begin{tikzcd} \mathbb{X} \arrow[dr, "\varphi"'] & & \mathbb{Y} \arrow[dl, "\psi"] \\ & \mathbb{W} & \end{tikzcd}$$

such that $\mathbb{X} \xrightarrow{\varphi} \mathbb{W}$ and $\mathbb{Y} \xrightarrow{\psi} \mathbb{W}$ are both weak equivalences of logical groupoids.

Appendix A

Elementary proofs for syntactic categories

This appendix is intended to supplement Section III.3 by providing entirely elementary proofs of those results therein that make use of the internal language of a doctrine. Explicating an elementary proof elucidates those parts where the structure of an existential doctrine is necessary.

Let $P\colon C^{\mathrm{op}} \to \mathbf{MSLat}$ be an existential doctrine. In Section III.3, we used the internal language of P to intuit the following results.

(1) For each arrow $(c, U) \xrightarrow{f} (d, V)$ of $C \rtimes P$,

$$\exists_{\mathrm{id}_c \times f} U \in P(c \times d)$$

defines a provably functional relation $(c, U) \to (d, V)$.

(2) Given composable arrows

$$(c, U) \xrightarrow{\ f\ } (d, V) \xrightarrow{\ g\ } (e, W)$$

in $C \rtimes P$, there is an equality

$$\exists_{\mathrm{id}_c \times g \circ f} U = \exists_{\mathrm{pr}_{1,3}}(P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c \times f} U \wedge P(\mathrm{pr}_{2,3})\exists_{\mathrm{id}_d \times g} V),$$

i.e. ζ^P preserves composites.

(3) If $W \in P(c \times d)$ defines a provably functional relation $(c, U) \xrightarrow{W} (d, V)$, then the composite of the pair

$$(c \times d, W) \xrightarrow{\exists_{\mathrm{id}_{c \times d} \times \mathrm{pr}_1} W} (c, U) \xrightarrow{\quad W \quad} (d, V)$$

is the provably functional relation $(c \times d, W) \xrightarrow{\exists_{\mathrm{id}_{c \times d} \times \mathrm{pr}_2} W} (d, V)$.

We provide an elementary proof of each in turn, without use of the internal language.

Lemma A.1. *Let $P\colon C^{\mathrm{op}} \to \mathbf{MSLat}$ be an existential doctrine. For each arrow $(c, U) \xrightarrow{f} (d, V)$ of $C \rtimes P$, i.e. whenever $U \leqslant P(f)(V)$, the proposition*

$$\exists_{\mathrm{id}_c \times f} U \in P(c \times d)$$

is a provably functional relation $(c, U) \to (d, V) \in \mathbf{Syn}(P)$.

Proof. We must check the three conditions from Definition III.32(ii) are satisfied. Using the inequalities $U \leqslant U = P(\mathrm{pr}_1 \circ \mathrm{id}_c \times f)(U)$ and $U \leqslant P(f)(V) = P(\mathrm{pr}_2 \circ \mathrm{id}_c \times f)(V)$, we obtain the first desired inequality

$$\begin{aligned} & U \leqslant P(\mathrm{pr}_1 \circ \mathrm{id}_c \times f)(U),\ P(\mathrm{pr}_2 \circ \mathrm{id}_c \times f)(V) \\ \Longrightarrow\ & U \leqslant P(\mathrm{id}_c \times f)P(\mathrm{pr}_1)(U),\ P(\mathrm{id}_c \times f)P(\mathrm{pr}_2)(V), \\ \Longrightarrow\ & \exists_{\mathrm{id}_c \times f}(U) \leqslant P(\mathrm{pr}_1)(U) \wedge P(\mathrm{pr}_2)(V). \end{aligned}$$

The second desired inequality,

$$P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{1,3})\exists_{\mathrm{id}_c\times f}U \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d,$$

is effectively transitivity of the internal equality predicate. We first note that all the squares in the diagram

$$\begin{tikzcd} d \arrow[d, "\Delta_d"'] \arrow[dr, phantom, "①"] & c \arrow[l, "f"'] \arrow[r, "\mathrm{id}_c\times f"] \arrow[d, "\mathrm{id}_c\times f"] \arrow[dr, phantom, "②"] & c\times d \arrow[r, "\mathrm{pr}_1"] \arrow[d, "(\mathrm{id}_c, f, \mathrm{id}_d)"] \arrow[dr, phantom, "③"] & c \arrow[d, "\mathrm{id}_c\times f"] \\ d\times d & c\times d \arrow[l, "(f,\mathrm{id}_d)"'] \arrow[r, "(\mathrm{id}_c,\mathrm{id}_d,f)"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "④"] & c\times d\times d \arrow[r, "\mathrm{pr}_{1,2}"] \arrow[d, "\mathrm{pr}_{1,3}"] & c\times d \\ & c \arrow[r, "\mathrm{id}_c\times f"] & c\times d & \end{tikzcd}$$

are pullbacks, where $(\mathrm{id}_c, f, \mathrm{id}_d)$ and $(\mathrm{id}_c, \mathrm{id}_d, f)$ denote the universally obtained maps

$$\begin{tikzcd} c \arrow[dd, bend right, "\mathrm{id}_c"'] \arrow[ddr, "f"] & c\times d \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, dashed, "(\mathrm{id}_c,f,\mathrm{id}_d)"] & d \arrow[dd, bend left, "\mathrm{id}_d"] \\ & c\times d\times d \arrow[dl, "\mathrm{pr}_1'"] \arrow[d, "\mathrm{pr}_2'"] \arrow[dr, "\mathrm{pr}_3'"] & \\ c & d & d, \end{tikzcd} \qquad \begin{tikzcd} c \arrow[dd, bend right, "\mathrm{id}_c"'] \arrow[ddrr, bend right, "f"] & c\times d \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, dashed, "(\mathrm{id}_c,\mathrm{id}_d,f)"] & d \arrow[dd, bend left, "\mathrm{id}_d"] \\ & c\times d\times d \arrow[dl, "\mathrm{pr}_1'"] \arrow[d, "\mathrm{pr}_2'"] \arrow[dr, "\mathrm{pr}_3'"] & \\ c & d & d. \end{tikzcd}$$

We note also that $(f, \mathrm{id}_d) = \mathrm{pr}_{2,3} \circ (\mathrm{id}_c, f, \mathrm{id}_d)$. Beginning with the identity inequality $\exists_{\mathrm{id}_c\times f}\top_c \leqslant \exists_{\mathrm{id}_c\times f}\top_c$, we conclude that

$$\begin{aligned} & \exists_{\mathrm{id}_c\times f}\top_c \leqslant \exists_{\mathrm{id}_c\times f}\top_c \\ \Longrightarrow\ & \exists_{\mathrm{id}_c\times f}P(\mathrm{id}_c \times f)(\top_{c\times d}) \leqslant \exists_{\mathrm{id}_c\times f}P(f)(\top_d), \\ \Longrightarrow\ & P(\mathrm{id}_c, f, \mathrm{id}_d)\exists_{(\mathrm{id}_c,\mathrm{id}_d,f)}\top_{c\times d} \leqslant P(f, \mathrm{id}_d)\exists_{\Delta_d}\top_d \quad \text{by ①, ② and B.-C.,} \\ \Longrightarrow\ & P(\mathrm{id}_c, f, \mathrm{id}_d)\exists_{(\mathrm{id}_c,\mathrm{id}_d,f)}\top_{c\times d} \leqslant P(\mathrm{id}_c, f, \mathrm{id}_d)P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d, \\ \Longrightarrow\ & \exists_{(\mathrm{id}_c,f,\mathrm{id}_d)}(\top_{c\times d} \wedge P(\mathrm{id}_c, f, \mathrm{id}_d)\exists_{(\mathrm{id}_c,\mathrm{id}_d,f)}\top_{c\times d}) \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d \\ \Longrightarrow\ & \exists_{(\mathrm{id}_c,f,\mathrm{id}_d)}\top_{c\times d} \wedge \exists_{(\mathrm{id}_c,\mathrm{id}_d,f)}\top_{c\times d} \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d \quad \text{by Frobenius,} \\ \Longrightarrow\ & \exists_{(\mathrm{id}_c,f,\mathrm{id}_d)}P(\mathrm{pr}_1)(\top_c) \wedge \exists_{(\mathrm{id}_c,\mathrm{id}_d,f)}P(\mathrm{pr}_1)(\top_c) \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d, \\ \Longrightarrow\ & P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}\top_c \wedge P(\mathrm{pr}_{1,3})\exists_{\mathrm{id}_c\times f}\top_c \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d \quad \text{by ③, ④ and B.-C.} \end{aligned}$$

We need now only note that $U \leqslant \top_c$ to achieve our desired inequality that

$$P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{1,3})\exists_{\mathrm{id}_c\times f}U \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d.$$

The final inequality is obtained via

$$U = \exists_{\mathrm{pr}_1\circ(\mathrm{id}_c\times f)}U = \exists_{\mathrm{pr}_1}\exists_{\mathrm{id}_c\times f}U.$$

□

Lemma A.2. *Given composable arrows*

$$(c, U) \xrightarrow{f} (d, V) \xrightarrow{g} (e, W)$$

in $C \rtimes P$, there is an equality

$$\exists_{\mathrm{id}_c\times g\circ f}U = \exists_{pr_{1,3}}(P(pr_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(pr_{2,3})\exists_{\mathrm{id}_d\times g}V),$$

i.e. the assignments

$$(c, U) \mapsto (c, U),$$
$$(c, U) \xrightarrow{f} (d, V) \mapsto (c, U) \xrightarrow{\exists_{\mathrm{id}_c\times f}U} (d, V)$$

define a functor $\zeta^P \colon C \rtimes P \to \mathbf{Syn}(P)$.

Proof. There is a pair of composable arrows

$$(c, U) \xrightarrow{f} (d, V) \xrightarrow{g} (e, W)$$

in $C \rtimes P$ if $U \leqslant P(f)(V)$ and $V \leqslant P(g)(W)$. The composite of the arrows

$$(c, U) \xrightarrow{\exists_{\mathrm{id}_c\times f}U} (d, V) \xrightarrow{\exists_{\mathrm{id}_d\times g}V} (e, W)$$

in $\mathbf{Syn}(P)$ is given by the predicate

$$\exists_{\mathrm{pr}_{1,3}}(P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{2,3})\exists_{\mathrm{id}_d\times g}V).$$

Thus, ζ^P preserves composites if we are able to prove that

$$\exists_{\mathrm{id}_c\times g\circ f}U = \exists_{\mathrm{pr}_{1,3}}(P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{2,3})\exists_{\mathrm{id}_d\times g}V).$$

To demonstrate this equality, we first note that all the squares in the diagram

$$\begin{tikzcd}
c \arrow[r, "\mathrm{id}_c\times f"] \arrow[d, "\mathrm{id}_c\times g\circ f"'] \arrow[dr, phantom, "①"] & c\times d \arrow[r, "\mathrm{pr}_2"] \arrow[d, "(\mathrm{id}_c{,}\mathrm{id}_d\times g)"] \arrow[dr, phantom, "③"] & d \arrow[d, "\mathrm{id}_d\times g"] \\
c\times e \arrow[r, "(\mathrm{id}_c\times f{,}\mathrm{id}_e)"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "②"] & c\times d\times e \arrow[r, "\mathrm{pr}_{2,3}"] \arrow[d, "\mathrm{pr}_{1,2}"] & d\times e \\
c \arrow[r, "\mathrm{id}_c\times f"] & c\times d &
\end{tikzcd}$$

are pullbacks. Therefore, we have a chain of equalities

$$\begin{aligned}
&\exists_{\mathrm{pr}_{1,3}}(P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{2,3})\exists_{\mathrm{id}_d\times g}V) \\
&= \exists_{\mathrm{pr}_{1,3}}(\exists_{(\mathrm{id}_c\times f,\mathrm{id}_e)}P(\mathrm{pr}_1)(U) \wedge \exists_{(\mathrm{id}_c,\mathrm{id}_d\times g)}P(\mathrm{pr}_2)(V)) && \text{by ②, ③ and B.-C.,}\\
&= \exists_{\mathrm{pr}_{1,3}}\exists_{(\mathrm{id}_c\times f,\mathrm{id}_e)}(P(\mathrm{pr}_1)(U) \wedge P((\mathrm{id}_c\times f,\mathrm{id}_e))\exists_{(\mathrm{id}_c,\mathrm{id}_d\times g)}P(\mathrm{pr}_2)(V)) && \text{by Frobenius,}\\
&= P(\mathrm{pr}_1)(U) \wedge P((\mathrm{id}_c\times f,\mathrm{id}_e))\exists_{(\mathrm{id}_c,\mathrm{id}_d\times g)}P(\mathrm{pr}_2)(V)), \\
&= P(\mathrm{pr}_1)(U) \wedge \exists_{\mathrm{id}_c\times g\circ f}P(\mathrm{id}_c\times f)P(\mathrm{pr}_2)(V) && \text{by ① and B.-C.,}\\
&= \exists_{\mathrm{id}_c\times g\circ f}(P(\mathrm{id}_c\times gf)P(\mathrm{pr}_1)(U) \wedge P(\mathrm{id}_c\times f)P(\mathrm{pr}_2)(V)) && \text{by Frobenius,}\\
&= \exists_{\mathrm{id}_c\times g\circ f}(U \wedge P(f)(V)), \\
&= \exists_{\mathrm{id}_c\times g\circ f}U && \text{since } U \leqslant P(f)(V).
\end{aligned}$$

Thus, we achieve the desired equality

$$\exists_{\mathrm{pr}_{1,3}}(P(\mathrm{pr}_{1,2})\exists_{\mathrm{id}_c\times f}U \wedge P(\mathrm{pr}_{2,3})\exists_{\mathrm{id}_d\times g}V) = \exists_{\mathrm{id}_c\times g\circ f}U.$$

We observe also that the identity arrow $(c,U) \xrightarrow{\mathrm{id}_c} (c,U)$ in $\mathcal{C} \rtimes P$ gets assigned to the identity arrow $(c,U) \xrightarrow{\exists_{\Delta_c}U} (c,U)$ in $\mathbf{Syn}(P)$. Hence, ζ^P is functorial. □

Lemma A.3. *If $(c,U) \xrightarrow{W} (d,V)$ is a provably functional relation, then the composite of the pair*

$$(c\times d, W) \xrightarrow{\exists_{\mathrm{id}_{c\times d}\times pr_1}W} (c,U) \xrightarrow{\quad W \quad} (d,V)$$

is the provably functional relation $(c\times d, W) \xrightarrow{\exists_{\mathrm{id}_{c\times d}\times pr_2}W} (d,V)$ and thus in the image of ζ^P.

Proof. If $(c,U) \xrightarrow{W} (d,V)$ is a provably functional relation, i.e. an arrow of $\mathbf{Syn}(P)$, then as $W \leqslant P(\mathrm{pr}_1)(U)$ there is an arrow $(c\times d, W) \xrightarrow{\mathrm{pr}_1} (c,U)$ of $\mathcal{C} \rtimes P$. The composite of the pair

$$(c\times d, W) \xrightarrow{\zeta^P(\mathrm{pr}_1)=\exists_{\mathrm{id}_{c\times d}\times \mathrm{pr}_1}W} (c,U) \xrightarrow{\quad W \quad} (d,V) \tag{A.a}$$

is given by $\exists_{\mathrm{pr}_{1,2,4}}(\mathrm{pr}_{1,2,3}\exists_{\mathrm{id}_{c\times d}\times \mathrm{pr}_1}(W) \wedge \mathrm{pr}_{2,4}(W))$. We wish to show that this arrow lies in the image of ζ^P, namely that there is an equality

$$\exists_{\mathrm{id}_{c\times d}\times \mathrm{pr}_2}W = \exists_{\mathrm{pr}_{1,2,4}}(\mathrm{pr}_{1,2,3}\exists_{\mathrm{id}_{c\times d}\times \mathrm{pr}_1}(W) \wedge \mathrm{pr}_{2,4}(W)).$$

We first gather the necessary observations we will need. Both the squares

$$\begin{tikzcd}
c\times d\times d \arrow[r, "\mathrm{pr}_{1,2}"] \arrow[d, "(\mathrm{id}_{c\times d}\times \mathrm{pr}_1{,}\mathrm{id}_d)"'] & c\times d \arrow[d, "\mathrm{id}_{c\times d}\times \mathrm{pr}_1"] \\
c\times d\times c\times d \arrow[r, "\mathrm{pr}_{1,2,3}"] & c\times d\times c
\end{tikzcd} \tag{A.b}$$

and

$$\begin{tikzcd}
c\times d \arrow[r, "\mathrm{id}_{c\times d}\times \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_2"'] & c\times d\times d \arrow[d, "\mathrm{pr}_{2,3}"] \\
d \arrow[r, "\Delta_d"] & d\times d
\end{tikzcd} \tag{A.c}$$

are pullbacks, and by definition there is the inequality

$$P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W) \leqslant P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d. \tag{A.d}$$

Therefore, there is a chain of equalities

$$\begin{aligned}
&\exists_{\mathrm{pr}_{1,2,4}}(P(\mathrm{pr}_{1,2,3})\exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_1}(W) \wedge P(\mathrm{pr}_{2,4})(W)) \\
&= \exists_{\mathrm{pr}_{1,2,4}}(\exists_{(\mathrm{id}_{c\times d}\times\mathrm{pr}_1,\mathrm{id}_d)}P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{2,4})(W)) && \text{by (A.b) and B.-C.,} \\
&= \exists_{\mathrm{pr}_{1,2,4}}\exists_{(\mathrm{id}_{c\times d}\times\mathrm{pr}_1,\mathrm{id}_d)}(P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{id}_{c\times d}\times\mathrm{pr}_1,\mathrm{id}_d)P(\mathrm{pr}_{2,4})(W)) && \text{by Frobenius,} \\
&= P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W), \\
&= P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W) \wedge P(\mathrm{pr}_{2,3})\exists_{\Delta_d}\top_d && \text{using (A.d),} \\
&= P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W) \wedge \exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_2}P(\mathrm{pr}_2)(\top_d) && \text{by (A.c) and B.-C.,} \\
&= P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{pr}_{1,3})(W) \wedge \exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_2}\top_{c\times d}, \\
&= \exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_2}(P(\mathrm{id}_{c\times d}\times\mathrm{pr}_2)P(\mathrm{pr}_{1,2})(W) \wedge P(\mathrm{id}_{c\times d}\times\mathrm{pr}_2)P(\mathrm{pr}_{1,3})(W) \wedge \top_{c\times d}) && \text{by Frobenius,} \\
&= \exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_2}(P(\mathrm{id}_{c\times d})(W) \wedge P(\mathrm{id}_{c\times d})(W)), \\
&= \exists_{\mathrm{id}_{c\times d}\times\mathrm{pr}_2}W.
\end{aligned}$$

Hence, we conclude that the composite of (A.a) is the image under ζ^P of the arrow

$$(c \times d, W) \xrightarrow{\mathrm{pr}_2} (d, V) \in C \rtimes P$$

as desired. □

Appendix B

Descent data and equivariant sheaves

In this appendix we explicitly spell out the equivalence between the datum of a compatible X_1-action on a local homeomorphism $q\colon Y \to X_0$ and descent datum (Y, θ) for a topological/localic groupoid $\mathbb{X}$. Thereby, we are free to using either definitions when discussing the topos of sheaves $\mathbf{Sh}(\mathbb{X})$. The equivalence is merely a case of unravelling definitions, but since this can at times be fiddly, we include an exposition here. We will argue in the language of point-set topology, but recall from Remark V.18 that this also demonstrates the equivalence for locales.

Given a local homeomorphism $q\colon Y \to X_0$ with a compatible X_1-action

$$\beta\colon Y \times_{X_0} X_1 \longrightarrow Y,$$

the corresponding descent datum is the pair (Y, θ_β) where θ_β is the induced map

$$\begin{tikzcd}
s^*(Y) \arrow[dr, dashed, "\theta_\beta"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right] & & \\
& t^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "q"] \\
& X_1 \arrow[r, "t"'] & X_0,
\end{tikzcd}$$

where the outside square commutes by the axiom $q(\beta(y, \alpha)) = t(\alpha)$ of β.

The spaces $s^*(Y)$ and $t^*(Y)$ are

$$s^*(Y) = \{(y, \alpha) \in Y \times X_1 \mid s(\alpha) = q(y)\},$$
$$t^*(Y) = \{(y, \alpha) \in Y \times X_1 \mid t(\alpha) = q(y)\},$$

and θ_β is the map which sends $(y, \alpha) \in s^*(Y)$ to $(\beta(y, \alpha), \alpha) \in t^*(Y)$. We first show that θ_β does indeed define descent datum on Y.

The condition $e^*(\theta_\beta) = \mathrm{id}_Y$ asserts that the map $e^*(\theta_\beta)$ in the composite pullback diagram below is canonically the identity on Y.

$$\begin{tikzcd} e^*s^*(Y) \arrow[r] \arrow[d, "e^*(\theta_\beta)"'] \arrow[dr, phantom, "\lrcorner", very near start] & s^*(Y) \arrow[d, "\theta_\beta"] \\ e^*t^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & t^*(Y) \arrow[d] \\ X_0 \arrow[r, "e"'] & X_1 \end{tikzcd}$$

The space $e^*s^*(Y)$ is given by

$$e^*s^*(Y) = \{(x, y, \alpha) \in X_0 \times Y \times X_1 \mid e(x) = \alpha,\ s(\alpha) = q(y)\}$$

and similarly

$$e^*t^*(Y) = \{(x, y, \alpha) \in X_0 \times Y \times X_1 \mid e(x) = \alpha,\ t(\alpha) = q(y)\}.$$

The map $e^*(\theta_\beta)\colon e^*s^*(Y) \to e^*t^*(Y)$ acts by

$$(x, y, \alpha) \mapsto (x, \beta(y, \alpha), \alpha).$$

But since $x = s(e(x)) = s(\alpha) = q(y)$, a triple $(x, y, \alpha) \in e^*s^*(Y)$ is entirely determined by y. Thus, there is a canonical homeomorphism $e^*s^*(Y) \cong Y$ given by projecting onto the second component of the tuple. Similarly, the same projection yields a homeomorphism $e^*t^*(Y) \cong Y$. Since $\alpha = e(q(y))$ for each $(x, y, \alpha) \in e^*s^*(Y)$, we observe that $\beta(y, \alpha) = \beta(y, e(q(y))) = y$. Thus, we have a commuting triangle

$$\begin{tikzcd} & e^*s^*(Y) \arrow[dl, "\sim"'] \arrow[dd, "e^*(\theta_\beta)"] \\ Y & \\ & e^*t^*(Y), \arrow[ul, "\sim"] \end{tikzcd}$$

as required.

Now we turn to the condition that $m^*(\theta_\beta) = \mathrm{pr}_2^*(\theta_\beta) \circ \mathrm{pr}_1^*(\theta_\beta)$. The spaces involved can be expressed as

$$\begin{aligned} \mathrm{pr}_1^*s^*(Y) &= \{(y, \alpha, \gamma) \in Y \times X_1 \times X_1 \mid s(\mathrm{pr}_1(\alpha, \gamma)) = s(\alpha) = q(y),\ t(\alpha) = s(\gamma)\}, \\ \mathrm{pr}_1^*t^*(Y) &= \{(y, \alpha, \gamma) \in Y \times X_1 \times X_1 \mid t(\mathrm{pr}_1(\alpha, \gamma)) = t(\alpha) = q(y),\ t(\alpha) = s(\gamma)\}, \\ \mathrm{pr}_2^*s^*(Y) &= \{(y, \alpha, \gamma) \in Y \times X_1 \times X_1 \mid s(\mathrm{pr}_2(\alpha, \gamma)) = s(\gamma) = q(y),\ t(\alpha) = s(\gamma)\}, \\ \mathrm{pr}_2^*t^*(Y) &= \{(y, \alpha, \gamma) \in Y \times X_1 \times X_1 \mid t(\mathrm{pr}_2(\alpha, \gamma)) = t(\gamma) = q(y),\ t(\alpha) = s(\gamma)\}. \end{aligned}$$

Using the equations $s \circ m = s \circ \mathrm{pr}_1$ and $t \circ m = t \circ \mathrm{pr}_2$, and the commutativity of the pullback square

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "t"'] & X_0, \end{tikzcd}$$

we conclude that

$$\begin{aligned} m^*s^*(Y) &= \mathrm{pr}_1^*s^*(Y), \\ m^*t^*(Y) &= \mathrm{pr}_2^*t^*(Y), \\ \mathrm{pr}_1^*t^*(Y) &= \mathrm{pr}_2^*s^*(Y) \end{aligned}$$

Thus, the equation $m^*(\theta_\beta) = \mathrm{pr}_2^*(\theta_\beta) \circ \mathrm{pr}_1^*(\theta_\beta)$, i.e.

$$\left[m^*s^*(Y) \xrightarrow{m^*(\theta_\beta)} m^*t^*(Y)\right] = \left[\mathrm{pr}_1^*s^*(Y) \xrightarrow{\mathrm{pr}_1^*(\theta_\beta)} \mathrm{pr}_1^*t^*(Y) = \mathrm{pr}_2^*s^*(Y) \xrightarrow{\mathrm{pr}_2^*(\theta_\beta)} \mathrm{pr}_2^*t^*(Y)\right],$$

type-checks.

The map $\mathrm{pr}_1^*(\theta_\beta)$ is the map in the double pullback

$$\begin{tikzcd} \mathrm{pr}_1^*s^*(Y) \arrow[r] \arrow[d, "\pi_1^*(\theta_\beta)"'] \arrow[dr, phantom, "\lrcorner"] & s^*(Y) \arrow[d, "\theta_\beta"] \\ \mathrm{pr}_1^*t^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & t^*(Y) \arrow[d] \\ X_1 \times_{X_0} X_1 \arrow[r, "\mathrm{pr}_1"] & X_1, \end{tikzcd}$$

and therefore acts by

$$(y, \alpha, \gamma) \mapsto (\beta(y, \alpha), \alpha, \gamma).$$

Similarly, $\mathrm{pr}_2^*s^*(Y) \xrightarrow{\mathrm{pr}_2^*(\theta_\beta)} \mathrm{pr}_2^*t^*(Y)$ acts by

$$(y, \alpha, \gamma) \mapsto (\beta(y, \gamma), \alpha, \gamma)$$

and $m^*s^*(Y) \xrightarrow{m^*(\theta_\beta)} m^*t^*(Y)$ acts by

$$(y, \alpha) \mapsto (\beta(y, m(\alpha, \gamma)), \alpha, \gamma).$$

Thus, we observe that

$$\begin{aligned} (\mathrm{pr}_2^*(\theta_\beta) \circ \mathrm{pr}_1^*(\theta_\beta))(y, \alpha, \gamma) &= \mathrm{pr}_2^*(\theta_\beta)(\beta(y, \alpha), \alpha, \gamma) \\ &= (\beta(\beta(y, \alpha), \beta), \alpha, \gamma) \\ &= (\beta(y, m(\alpha, \gamma)), \alpha, \gamma) \\ &= m^*(\theta_\beta)(y, \alpha, \gamma). \end{aligned}$$

Hence, the pair (Y, θ_β) indeed constitutes descent datum.

An equivariant map $Y \xrightarrow{f} Y'$ between spaces with respective X_1-actions β and β' also constitutes a morphism of descent data $(Y, \beta) \xrightarrow{f} (Y', \beta')$. The required commutativity condition,

$$t^*(f) \circ \theta_\beta = \theta_{\beta'} \circ s^*(f),$$

is forced by universal property of $t^*(Y')$ in the commutative diagram

$$\begin{tikzcd}
s^*(Y) \arrow[rr, "s^*(f)"] \arrow[d, "\beta"'] \arrow[dr, "\theta_\beta"] & & s^*(Y') \arrow[d, "\beta'"'] \arrow[dr, "\theta_{\beta'}"] & \\
Y \arrow[rr, "f"] \arrow[dd] & t^*(Y) \arrow[l] \arrow[rr, "t^*(f)"] \arrow[dd] & Y' \arrow[dd] & t^*(Y') \arrow[l] \arrow[dd] \\
& & & \\
X_0 \arrow[rr, equal] & & X_0 & \\
& X_1 \arrow[ul, "t"] \arrow[rr, equal] & & X_1 . \arrow[ul, "t"]
\end{tikzcd}$$

For the other direction, suppose we are given a descent datum (Y, θ). We then obtain a compatible X_1-action $\beta_\theta \colon s^*(Y) \to Y$ by taking β_θ to be the composite

$$Y \times_{X_0} X_1 \cong s^*(Y) \xrightarrow{\theta} t^*(Y) \xrightarrow{\mathrm{pr}_1} Y.$$

Checking that β_θ defines a legitimate X_1-action, or that a morphism of descent data $(Y, \theta) \xrightarrow{f} (Y', \theta')$ yields an equivariant map $(Y, \beta_\theta) \xrightarrow{f} (Y', \beta_{\theta'})$, essentially amounts to the reverse of what we have done above, and so we omit the details. Finally, note that the two correspondences are mutual inverses since, for all $(y, \alpha) \in Y \times_{X_0} X_1$,

$$\beta_{\theta_\beta}(y, \alpha) = \mathrm{pr}_1(\beta(y, \alpha), \alpha) = \beta(y, \alpha)$$

and similarly $\theta_{\beta_\theta} = \theta$.

Concluding remarks

Relative topos-theoretic techniques in doctrine theory

In Part A, we saw how the techniques of relative topos theory can be successfully applied to the study of completions of doctrines. Broadly speaking, there are two kinds of completions of doctrines considered in the literature. There are those completions which add structure to the fibres, such as Trotta's existential completion [119] or Coumans' canonical completion [30], and then there are those which add structure to the indexing category, exemplified in the work [81]. The geometric completion considered in Chapter IV belongs to the first species, but a continuation of the work commenced in this dissertation encompasses the latter as well.

Exact completions of doctrines. As observed in [119, §6], by composing the pseudo-adjunctions coming from the existential completion, the syntactic category construction and the exact completion of a regular category (see [28, §2.3]), as in the diagram

$$\begin{tikzcd} \mathbf{PrimDoc} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{ExDoc} \arrow[l, hook', shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Reg} \arrow[l, hook', shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Exact}, \arrow[l, hook', shift left] \end{tikzcd}$$

we obtain the *exact completion of a primary doctrine* in the sense of [83]. The work of Maietti, Pasquali and Rosolini [83], [80] has been fundamental in understanding the exact completion of a doctrine via a series of doctrinal completions. The composite functor $\mathbf{PER} \colon \mathbf{ExDoc} \to \mathbf{Exact}$ is the so-called *'tripos' construction* (see [53], [100]), also called the partial equivalence relation construction since the objects of the resultant category are *partial equivalence relations* in the internal language of the doctrine.

Since the geometric completion of a primary doctrine interprets geometric logic, we could also consider taking the analogous category whose objects are finite, or indeed infinite, tuples of internal partial equivalence relations. In the case of finite tuples, this would obtain a doctrinal version of the *pretopos completion* from [87, §8.4]. More generally, we recover a whole spectrum of 'tripos-like' completions for doctrines, akin to the analogous exactness completions for categories (see [107], [115]). Not only are such 'tripos-like' constructions employed to construct notable examples of elementary topoi, such as Hyland's *effective topos*, but their connection to the model-theoretic *Shelah's elimination of imaginaries* construction has also been noted (see [48]).

While the infinite partial equivalence relation construction for a (primary) doctrine is only 'syntactically parsable' when the doctrine also interprets geometric logic, this restriction can be evaded by first taking the geometric completion. The unifying role of the geometric completion in relating the various 'tripos-like' constructions will be the subject of future work.

Removing hypotheses. Thus, our approach parallels the burgeoning interest in describing exact completions for weaker and weaker initial structures. For example, the exact completion of a category with finite limits given in [27] is extended by Carboni and Vitale in [28] to describe the exact completion of a category with only *weak* finite limits. This interest has been paralleled by recent work for doctrines, e.g. in [29].

Our use of relative topos theory permitted a construction of the geometric completion at the extreme end of this spectrum in that it can be defined for entirely *unstructured* doctrines (that is, any **PreOrd**-valued pseudo-functor). Since morphisms of (relative) sites 'preserve finite limit data relatively' (see Remark I.4), a relative topos theoretic approach to exactness completions will similarly facilitate the construction of exact completions with only the weakest viable hypotheses.

Representing groupoids and Morita equivalence

Our original motivation in pursuing a marriage of topology and predicate logic was to furnish the tools needed for a further investigation of Morita equivalences. Recall that two theories $\mathbb{T}, \mathbb{T}'$ with classifying topoi are *Morita equivalent* if there is an equivalence of topoi $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$, or equivalently if there is an equivalence of model categories

$$\mathbb{T}\text{-mod}(\mathcal{F}) \simeq \mathbb{T}'\text{-mod}(\mathcal{F}),$$

natural in any topos $\mathcal{F}$. From the point of view of the category theorist, for whom equivalence replaces identity, a Morita equivalence between theories ought to be 'trivial' in some sense. However, superficially this is far from the case. Just as with bi-interpretability, every theory is Morita equivalent to infinitely many other theories, and syntactically these theories can be very different (e.g., a single-sorted theory can be Morita equivalent to a multi-sorted theory, see Remark VI.2). Indeed, such equivalences are at the heart of Caramello's theory of 'topos-theoretic bridges' (see [22, §2.2]).

We proposed the use of topological groupoids as a syntax-invariant perspective on Morita equivalence, intending to translate the problem of identifying Morita equivalences of theories into one of topological algebra, with the additional benefit that the working mathematician often has a firmer grasp on a spotlighted tomogram of models of a theory than its entire syntax.

To conclude, we evaluate the efficacy of our contributions in respect to this goal. The omnipresent time restraints of a doctorate have also precluded some extensions to our study, which we highlight as potential avenues for future research.

A topological description of weak equivalences. Consider a weak equivalence $\mathbb{X} \xrightarrow{\varphi} \mathbb{Y}$ of logical groupoids. As currently formulated in Definition VIII.10, that φ is a weak equivalence relies on an oracular choice of theory simultaneously classified by the topoi $\mathbf{Sh}(\mathbb{X})$ and $\mathbf{Sh}(\mathbb{Y})$. To be considered a complete translation of the logical problem of Morita equivalence into one of topological algebra, an entirely topological characterisation of weak equivalences is required.

The *Moerdijk site* for $\mathbf{Sh}(\mathbb{X})$ (see [92, Definition 6.1]) provides a promising source of such a characterisation. Let $\mathbb{X}$ be a logical groupoid. The choice of a theory $\mathbb{T}$

over a signature Σ classified by $\mathbf{Sh}(\mathbb{X})$ corresponds to a choice of *definable sheaves* $(\llbracket \vec{x} : \varphi \rrbracket_{\mathbb{X}}, \pi_{\llbracket \vec{x}:\varphi \rrbracket}, \theta_{\llbracket \vec{x}:\varphi \rrbracket})$. We can therefore eliminate the need for a choice of definable sheaves by instead considering a suitable topologically defined set of sheaves. We note that the objects of the Moerdijk site are characterised in entirely topological terms and, moreover, it can be demonstrated that for any choice of theory classified by $\mathbf{Sh}(\mathbb{X})$, there exists a generating set of objects for $\mathbf{Sh}(\mathbb{X})$ that are definable sheaves contained in the Moerdijk site (cf. [5, Lemma 2.1.5]). The properties of conservativity and elimination of parameters ought thus to be translated into properties of the sheaves in the theory-invariant Moerdijk site.

Non-invertible 2-cells. When constructing the biequivalence

$$[\mathfrak{W}^{-1}]\mathbf{LogGrpd} \simeq \mathbf{Topos}^{\mathrm{iso}}_{w.e.p.}$$

in Theorem VIII.14, we were required to restrict our investigation to the invertible 2-cells of the bicategory $\mathbf{Topos}_{w.e.p.} \subseteq \mathbf{Topos}$ since a transformation between homomorphisms of topological groupoids is necessarily invertible. However, the slogan that representing groupoids possess 'sufficient information to recover' their topoi of sheaves suggests that the groupoids $\mathbb{X}, \mathbb{Y}$ do contain knowledge, in some fashion, of the non-invertible 2-cells

$$\begin{tikzcd} \mathbf{Sh}(\mathbb{X}) \arrow[r, bend left=50, "\mathbf{Sh}(\varphi)"] \arrow[r, bend right=50, "\mathbf{Sh}(\psi)"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Sh}(\mathbb{Y}), \end{tikzcd}$$

for any pair of parallel homomorphisms of topological groupoids φ, ψ.

As suggested by the analysis of [118, §16], [51, §5], *Sierpiński-valued homotopies* offer a potential solution. Recall that, in common parlance, a *homotopy* $f \overset{H}{\Rightarrow} g$ between parallel continuous maps $f, g\colon X \rightrightarrows Y$ is a continuous map $H\colon [0,1]\colon X \to Y$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$. A Sierpiński-valued homotopy is the same basic concept where the interval $[0,1]$ is replaced by the Sierpiński space $\mathbb{S}$. Such homotopies can be generalised, as in [51, Definition 5.1], to the setting of topological groupoids. Importantly, Sierpiński-valued homotopies are not invertible. This is the subject of on-going work with Graham Manuell.

Topos-theoretic invariants of topological groupoids. Given a theory $\mathbb{T}$ with a representing model groupoid $\mathbb{X}$, we can restrict which theories can be Morita equivalent to $\mathbb{T}$ by identifying topological properties of the topological groupoid $\mathbb{X}^{\tau\text{-log}_1}_{\tau\text{-log}_0}$ that are preserved under weak equivalences. These will correspond to topos-theoretic invariants of the classifying topos $\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$, and thereby contribute to the theory of 'topos-theoretic bridges' from [22, §2.2].

We have highlighted some syntactic properties already: in Proposition VII.35 and Proposition VII.45. The literature abounds with other examples, such as the aforementioned link between étale groupoids and étendues (see [3, §VI.9.8.2(e)]). Given the equivariant nature of the groupoid representation of logical theories, it is natural to expect the tools of algebraic topology to generate fruitful applications to logic. Steps have already been taken in this direction (see [16] and [15], for instance). The benefit of such a development can be bidirectional: both [65], [66] and [69] employ logically inspired intuition in the study of the cohomology theory of topoi.

Acknowledgements

A certain midshipman once explained to me the difference between 'type 1' fun and 'type 2' fun. The former is that which is fun intrinsically. The latter is that kind of fun which is best appreciated when reflecting after the fact, preferably over a hot beverage, the catharsis after a rain-sodden yomp being the archetypal example. A doctorate is, without doubt, 'type 2' fun.

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Bibliography

[1] S. Abramsky, "Domain theory and the logic of observable properties," Ph.D. dissertation, Queen Mary College, University of London, 1987. arXiv: `1112.0347 [cs.LO]`.

[2] G. Ahlbrandt and M. Ziegler, "Quasi finitely axiomatizable totally categorical theories," *Annals of Pure and Applied Logic*, vol. 30, no. 1, pp. 63–82, 1986.

[3] M. Artin, A. Grothendieck, and J. Verdier, *Théorie des Topos et Cohomologie Etale des Schémas*, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4) (Lecture Notes in Mathematics 305). Springer Berlin Heidelberg, 1973.

[4] S. Awodey, "Sheaf representations and duality in logic," in *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics*, C. Casadio and P. J. Scott, Eds. Springer International Publishing, 2021, pp. 39–57.

[5] S. Awodey and H. Forssell, "First-order logical duality," *Annals of Pure and Applied Logic*, vol. 164, no. 3, pp. 319–348, 2013.

[6] M. Barr, "Exact categories," in *Exact Categories and Categories of Sheaves*. Springer Berlin Heidelberg, 1971, pp. 1–120.

[7] T. W. Barrett and H. Halvorson, "Morita equivalence," *The Review of Symbolic Logic*, vol. 9, no. 3, pp. 556–582, 2016.

[8] L. Bartoli and O. Caramello, "On morphisms of relative toposes," 2023. arXiv: `2310.20691 [math.AG]`.

[9] I. Ben Yaacov, "Reconstruction of non-$\aleph_0$-categorical theories," *The Journal of Symbolic Logic*, vol. 87, no. 1, pp. 159–187, 2022.

[10] J. Bénabou, "Introduction to bicategories," in *Reports of the Midwest Category Seminar*, Springer Berlin Heidelberg, 1967, pp. 1–77.

[11] A. Blass and A. Ščedrov, "Boolean classifying topoi," *Journal of Pure and Applied Algebra*, vol. 28, no. 1, pp. 15–30, 1983.

[12] W. Block and D. Pigozzi, "Algebraizable logics," *Memoirs of the American Mathematical Society*, no. 396, 1989.

[13] F. Borceux, *Handbook of Categorical Algebra* (Encyclopedia of Mathematics and its Applications). Cambridge University Press, 1994.

[14] F. Borceux and G. Janelidze, *Galois Theories* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 2001.

[15] S. Breiner, "Scheme representation for first-order logic," Ph.D. dissertation, Carnegie Mellon University, 2014. [Online]. Available: `https://www.andrew.cmu.edu/user/awodey/students/breiner.pdf`.

[16] C. Butz, "Logical and cohomological aspects of the space of points of a topos," Ph.D. dissertation, University of Utrecht, 1996.

[17] C. Butz and I. Moerdijk, "Representing topoi by topological groupoids," *Journal of Pure and Applied Algebra*, vol. 130, no. 3, pp. 223–235, 1998.

[18] O. Caramello, "Fraïssé's construction from a topos-theoretic perspective," *Logica Universalis*, vol. 8, pp. 261–281, 2014.

[19] O. Caramello, "A topos-theoretic approach to Stone-type dualities," 2011. arXiv: `1103.3493 [math.CT]`.

[20] O. Caramello, "Atomic toposes and countable categoricity," *Applied Categorical Structures*, vol. 20, pp. 379–391, 2012.

[21] O. Caramello, "Topological Galois theory," *Advances in Mathematics*, vol. 291, pp. 646–695, 2016.

[22] O. Caramello, *Theories, sites, toposes: relating and studying mathematical theories through topos-theoretic 'bridges'*. Oxford University press, 2018.

[23] O. Caramello, "Denseness conditions, morphisms and equivalences of toposes," 2020. arXiv: `1906.08737 [math.CT]`.

[24] O. Caramello, "Fibred sites and existential toposes," 2022. arXiv: `2212.11693 [math.AG]`.

[25] O. Caramello and P. T. Johnstone, "De Morgan's law and the theory of fields," *Advances in Mathematics*, vol. 222, no. 6, pp. 2145–2152, 2009.

[26] O. Caramello and R. Zanfa, "Relative topos theory via stacks," 2021. arXiv: `2107.04417 [math.AG]`.

[27] A. Carboni, "Some free constructions in realizability and proof theory," *Journal of Pure and Applied Algebra*, vol. 103, no. 2, pp. 117–148, 1995.

[28] A. Carboni and E. Vitale, "Regular and exact completions," *Journal of Pure and Applied Algebra*, vol. 125, no. 1, pp. 79–116, 1998.

[29] C. J. Cioffo, "Biased elementary doctrines and quotient completions," 2023. arXiv: `2304.03066 [math.CT]`.

[30] D. Coumans, "Generalising canonical extension to the categorical setting," *Annals of Pure and Applied Logic*, vol. 163, no. 12, pp. 1940–1961, 2012.

[31] B. Davey and H. Priestley, *Introduction to Lattices and Order* (Cambridge mathematical textbooks). Cambridge University Press, 2002.

[32] R. Diaconescu, "Change of base for toposes with generators," *Journal of Pure and Applied Algebra*, vol. 6, no. 3, pp. 191–218, 1975.

[33] R. Diaconescu, "Grothendieck toposes have boolean points – a new proof," *Communications in Algebra*, vol. 4, pp. 723–729, 8 1976.

[34] E. J. Dubuc, "Localic Galois theory," *Advances in Mathematics*, vol. 175, no. 1, pp. 144–167, 2003.

[35] M. Escardó, "Joins in the frame of nuclei," *Applied Categorical Structures*, vol. 11, pp. 117–124, 2003.

[36] H. Forssell, "First-order logical duality," Ph.D. dissertation, Carnegie Mellon University, 2008.

[37] H. Forssell, "Topological representation of geometric theories," *Mathematical Logic Quarterly*, vol. 58, no. 6, pp. 380–393, 2012.

[38] J. Frey, "A fibrational study of realizability toposes," Ph.D. dissertation, Paris 7 University, 2013. arXiv: `1403.3672 [math.CT]`.

[39] P. J. Freyd and A. Scredrov, *Categories, Allegories*. North Holland Publishing Company, 1991.

[40] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*. Springer-Verlag, New York, 1967.

[41] R. Garner, "Ionads," *Journal of Pure and Applied Algebra*, vol. 216, no. 8, pp. 1734–1747, 2012.

[42] J. Giraud, "Classifying topos," in *Toposes, Algebraic Geometry and Logic*, F. W. Lawvere, Ed., Springer Berlin Heidelberg, 1972, pp. 43–56.

[43] S. van Gool and J. Marquès, "On duality and model theory for polyadic spaces," 2022. arXiv: `2210.01018 [math.LO]`.

[44] A. Grothendieck and M. Raynaud, *Revêtements Ètale et Groupe Fondamental*, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 1) (Lecture Notes in Mathematics 224). Springer Berlin Heidelberg, 1971.

[45] A. Grothendieck, "Sur quelques points d'algèbre homologique, I," *Tohoku Mathematical Journal*, vol. 9, no. 2, pp. 119–221, 1957.

[46] F. Guffanti, "Rich doctrines and Henkin's theorem," 2023. arXiv: `2310.08374 [math.CT]`.

[47] P. R. Halmos, "Polyadic boolean algebras," *Proceedings of the National Academy of Sciences*, vol. 40, no. 5, pp. 296–301, 1954.

[48] V. Harnik, "Model theory vs. categorical logic: Two approaches to pretopos completion (a.k.a. T^{eq})," in *Models, Logics, and Higher-Dimensional Categories: A Tribute to the Work of Mihály Makkai*, ser. CRM Proceedings and Lecture Notes, B. Hart, T. G. Kucera, A. Pillay, P. J. Scott, and R. A. G. Seely, Eds., vol. 53, American Mathematical Society, Providence, RI, 2011, pp. 79–106.

[49] R. Heckmann, "Spatiality of countably presentable locales (proved with the baire category theorem)," *Mathematical Structures in Computer Science*, vol. 25, no. 7, pp. 1607–1625, 2015.

[50] L. Henkin and A. Tarski, "Cylindric algebras," in *Lattice theory* (Proceedings of Symposia in Pure Mathematics), R. P. Dilworth, Ed., Proceedings of Symposia in Pure Mathematics. American Mathematical Society, Providence, RI, 1961, vol. 2, pp. 83–113.

[51] S. Henry and C. Townsend, "A classifying groupoid for compact hausdorff locales," 2023. arXiv: `2310.07785 [math.CT]`.

[52] W. Hodges, *Model Theory* (Encyclopedia of Mathematics and its Applications). Cambridge University Press, 1993.

[53] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts, "Tripos theory," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 88, no. 2, pp. 205–232, 1980.

[54] J. R. Isbell, "Atomless parts of spaces," *Mathematica Scandinavica*, vol. 31, no. 1, pp. 5–32, 1972.

[55] B. Jacobs, *Categorical logic and type theory* (Studies in Logic and the Foundations of Mathematics). Elsevier, 1999.

[56] P. Johnstone, "Tychonoff's theorem without the axiom of choice," *Fundamenta Mathematicae*, vol. 113, no. 1, pp. 21–35, 1981.

[57] P. T. Johnstone, *Topos theory*. Academic Press, London, 1977.

[58] P. T. Johnstone, "Open maps of toposes," *Manucripta Mathematica*, vol. 31, pp. 217–247, 1980.

[59] P. T. Johnstone, "Factorization theorems for geometric morphisms, I," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 22, no. 1, pp. 3–17, 1981.

[60] P. T. Johnstone, *Stone Spaces* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 1982.

[61] P. T. Johnstone, "The point of pointless topology," *Bulletin (New Series) of the American Mathematical Society*, vol. 8, no. 1, pp. 41–53, 1983.

[62] P. T. Johnstone, "Elements of the history of locale theory," in *Handbook of the History of General Topology*, C. E. Aull and R. Lowen, Eds. Dordrecht: Springer Netherlands, 2001, pp. 835–851.

[63] P. T. Johnstone, *Sketches of an Elephant: A topos theoretic compendium*. Oxford University Press, 2002.

[64] A. Joyal, "Polyadic spaces and elementary theories," in *Notices of the American Mathematical Society*, vol. 18, 1971, p. 563. [Online]. Available: `https://www.ams.org/journals/notices/197104/197104FullIssue.pdf`.

[65] A. Joyal and I. Moerdijk, "Toposes are cohomologically equivalent to spaces," *American Journal of Mathematics*, vol. 112, pp. 87–95, 1990.

[66] A. Joyal and I. Moerdijk, "Toposes as homotopy groupoids," *Advances in Mathematics*, vol. 80, no. 1, pp. 22–38, 1990.

[67] A. Joyal and R. Street, "Pullbacks equivalent to pseudopullbacks," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 34, no. 2, pp. 153–156, 1993.

[68] A. Joyal and M. Tierney, "An extension of the Galois theory of Grothendieck," *Memoirs of the American Mathematical Society*, vol. 51, no. 309, 1984.

[69] A. Joyal and G. Wraith, "Eilenberg-mac lane toposes and cohomology," in *Mathematical Applications of Category Theory*, ser. Contemporary Mathematics, J. W. Gray, Ed., vol. 30, American Mathematical Society, Providence, RI, 1984, pp. 117–131.

[70] M. Kamsma, "Type space functors and interpretations in positive logic," *Archive for Mathematical Logic*, vol. 62, pp. 1–28, 2023.

[71] P. Karazeris, "Notions of flatness relative to a Grothendieck topology," *Theory and Applications of Categories*, vol. 12, no. 5, pp. 225–236, 2004.

[72] G. Kelly and S. Lack, "On property-like structures," *Theory and Applications of Categories*, vol. 3, pp. 213–250, 1997.

[73] A. Kock and G. Reyes, "Doctrines in categorical logic," in *Handbook of mathematical logic*, ser. Studies in Logic and the Foundations of Mathematics, J. Barwise, Ed., vol. 90, Elsevier, 1977, pp. 283–313.

[74] A. Kock, "Monads for which structures are adjoint to units," *Journal of Pure and Applied Algebra*, vol. 104, no. 1, pp. 41–59, 1995.

[75] A. Kock and I. Moerdijk, "Presentations of étendues," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 32, no. 2, pp. 145–164, 1991.

[76] S. Lack, "A 2-categories companion," in *Towards Higher Categories*, J. C. Baez and J. P. May, Eds., New York: Springer, 2010, pp. 105–191.

[77] F. W. Lawvere, "Adjointness in foundations," *Dialectica*, vol. 23, no. 3/4, pp. 281–296, 1969.

[78] F. W. Lawvere, "Equality in hyperdoctrines and comprehension schema as an adjoint functor," in ser. New York Symposium on Application of Categorical Algebra, A. Heller, Ed., vol. 2, American Mathematical Society, 1970, pp. 1–14.

[79] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (Universitext). Springer New York, 1994.

[80] M. E. Maietti, F. Pasquali, and G. Rosolini, "Triposes, exact completions, and Hilbert's ϵ-operator," *Tbilisi Mathematical Journal*, vol. 10, no. 3, pp. 141–166, 2017.

[81] M. E. Maietti and G. Rosolini, "Elementary quotient completion," *Theory and Applications of Categories*, vol. 27, no. 17, pp. 445–463, 2013.

[82] M. E. Maietti and G. Rosolini, "Quotient completion for the foundation of constructive mathematics," *Logica Universalis*, vol. 7, pp. 371–402, 2013.

[83] M. E. Maietti and G. Rosolini, "Unifying exact completions," *Applied Categorical Structures*, vol. 23, pp. 43–52, 2015.

[84] M. E. Maietti and G. Rosolini, "Relating quotient completions via categorical logic," in *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, D. Probst and P. Schuster, Eds. Berlin, Boston: De Gruyter, 2016, pp. 229–250.

[85] M. Makkai, "Stone duality for first order logic," *Advances in Mathematics*, vol. 65, no. 2, pp. 97–170, 1987.

[86] M. Makkai, "Duality and definability in first order logic," *Memoirs of the American Mathematical Society*, vol. 105, no. 503, 1993.

[87] M. Makkai and G. E. Reyes, *First Order Categorical Logic: Model-Theoretical Methods in the Theory of Topoi and Related Categories* (Lecture Notes in Mathematics). Springer Berlin, Heidelberg, 1977.

[88] G. Manuell and J. L. Wrigley, "The representing localic groupoid for a geometric theory," 2023. arXiv: `2305.15209 [math.CT]`.

[89] D. Marker, *Model Theory: An Introduction* (Graduate Texts in Mathematics). Springer New York, NY, 2002.

[90] J. Milnor, "Construction of universal bundles, II," *Annals of Mathematics*, vol. 63, no. 3, pp. 430–436, 1956.

[91] I. Moerdijk, "Morita equivalence for continuous groups," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 103, no. 1, pp. 97–115, 1988.

[92] I. Moerdijk, "The classifying topos of a continuous groupoid, I," *Transactions of the American Mathematical Society*, vol. 310, pp. 629–668, 1988.

[93] I. Moerdijk, "The classifying topos of a continuous groupoid, II," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 31, no. 2, pp. 137–168, 1990.

[94] I. Moerdijk and D. Pronk, "Orbifolds, sheaves and groupoids," *K-Theory*, vol. 12, pp. 3–21, 1997.

[95] K. Morita, "Duality for modules and its applications to the theory of rings with minimum condition," *Science Reports of the Tokyo Kyoiku Daigaku, Section A*, vol. 6, no. 150, pp. 83–142, 1958.

[96] F. Pasquali, "A co-free construction for elementary doctrines," *Applied Categorical Structures*, vol. 23, pp. 29–41, 2015.

[97] J. Picado and A. Pultr, *Frames and Locales: Topology without points*. Springer Basel, 2012.

[98] A. M. Pitts, "The theory of triposes," Ph.D. dissertation, Cambridge University, 1981. [Online]. Available: `https://www.cl.cam.ac.uk/~amp12/papers/thet/thet.pdf`.

[99] A. M. Pitts, "An application of open maps to categorical logic," *Journal of Pure and Applied Algebra*, vol. 29, no. 3, pp. 313–326, 1983.

[100] A. M. Pitts, "Tripos theory in retrospect," *Mathematical Structures in Computer Science*, vol. 12, pp. 265–279, 2002.

[101] B. Poizat, "Une théorie de Galois imaginaire," *The Journal of Symbolic Logic*, vol. 48, no. 4, pp. 1151–1170, 1983.

[102] D. A. Pronk, "Etendues and stacks as bicategories of fractions," *Compositio Mathematica*, vol. 102, no. 3, pp. 243–303, 1996.

[103] W. Rether, *The étale topos of a scheme is the classifying topos of...?* MathStackExchange: `https://math.stackexchange.com/q/2728171`, 2018.

[104] M. Rogers, "Toposes of monoid actions," Ph.D. dissertation, University of Insubria, 2021. arXiv: `2112.10198 [math.CT]`.

[105] I. Satake, "On a generalization of the notion of manifold," *Proceedings of the National Academy of Sciences*, vol. 42, no. 6, pp. 359–363, 1956.

[106] R. A. G. Seely, "Hyperdoctrines, natural deduction and the Beck condition," *Mathematical Logic Quarterly*, vol. 29, no. 10, pp. 505–542, 1983.

[107] M. Shulman, "Exact completions and small sheaves," *Theory and Applications of Categories*, vol. 27, no. 7, pp. 97–173, 2012.

[108] H. Simmons, "Near-discreteness of modules and spaces as measured by Gabriel and Cantor," *Journal of Pure and Applied Algebra*, vol. 56, no. 2, pp. 119–162, 1989.

[109] B. Steinberg, *Toposes (topoi) as classifying toposes of groupoids*, MathOverflow: `https://mathoverflow.net/q/84197`, 2015.

[110] A. Stieglitz, "Equivariant sheaf cohomology," *Manuscripta Mathematica*, vol. 26, no. 3, pp. 201–221, 1978.

[111] M. H. Stone, "The theory of representation for Boolean algebras," *Transactions of the American Mathematical Society*, vol. 40, no. 1, pp. 37–111, 1936.

[112] M. H. Stone, "Topological representations of distributive lattices and Brouwerian logics," *Časopis pro pěstování matematiky a fysiky*, vol. 67, no. 1, pp. 1–25, 1938.

[113] R. Street, "Fibrations in bicategories," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 21, no. 2, pp. 111–160, 1980.

[114] R. Street, "Conspectus of variable categories," *Journal of Pure and Applied Algebra*, vol. 21, pp. 307–338, 1981.

[115] R. Street, "The family approach to total cocompleteness and toposes," *Transations of the American Mathematical Society*, vol. 284, no. 1, pp. 355–369, 1984.

[116] I. Tomašić, "A topos-theoretic view of difference algebra," 2020. arXiv: `2001.09075 [math.AG]`.

[117] C. Townsend, *Given a grothendieck topos, what does its localic groupoid look like?* MathOverflow: `https://mathoverflow.net/q/102652`, 2017.

[118] C. Townsend, "Hilsum–Skandalis maps as Frobenius adjunctions with application to geometric morphisms," *Tbilisi Mathematical Journal*, vol. 10, no. 3, pp. 83–120, 2017.

[119] D. Trotta, "The existential completion," *Theory and Applications of Categories*, vol. 35, pp. 1576–1607, 2020.

[120] D. Trotta, "An algebraic approach to the completions of elementary doctrines," 2021. arXiv: `2108.03415 [math.CT]`.

[121] D. Trotta and M. Spadetto, "Quantifier completions, choice principles and applications," 2021. arXiv: `2010.09111 [math.CT]`.

[122] D. Tsementzis, "A syntactic characterization of Morita equivalence," *The Journal of Symbolic Logic*, vol. 82, no. 4, pp. 1181–1198, 2017.

[123] S. Vickers, "Locales and toposes as spaces," in *Handbook of spatial logics*, M. Aiello, I. Pratt-Hartmann, and J. Van Benthem, Eds., Dordrecht: Springer, 2007, pp. 429–496.

[124] S. Vickers, "Continuity and geometric logic," *Journal of Applied Logic*, vol. 12, no. 1, pp. 14–27, 2014.

[125] J. L. Wrigley, "The logic and geometry of localic morphisms," talk at *Toposes Online*, Jun. 2021. [Online]. Available: `https://aroundtoposes.com/wp-content/uploads/2021/07/WrigleySlidesToposesOnline.pdf`.

[126] J. L. Wrigley, "On topological groupoids that represent theories," 2023. arXiv: `2306.16331 [math.CT]`.

[127] J. L. Wrigley, "Some properties of internal locale morphisms externalised," 2023. arXiv: `2301.00961 [math.AG]`.

[128] J. L. Wrigley, "The geometric completion of a doctrine," 2023. arXiv: `2304.07539 [math.CT]`.